$x^2 = \frac{1}{x+1}$ řešení? převod na "pevný bod"

2 iterace, $x_0 = 0$

$\exists$ řeš $\Rightarrow \frac{1}{x+1} \geq 0 \Rightarrow x > -1$

	L	P
$x = (-1)^+$	1	$< \infty$
$x = 0$	0	< 1
$x = 1$	1	$> \frac{1}{2}$
	$\downarrow$ ∞	$\downarrow$ 0

• standard

$\underbrace{x^2 - \frac{1}{x+1}}_{+x} = \underbrace{0}_{+x}$ || $f(x) = x^2 - \frac{1}{x+1}$

$\overbrace{x^2 + x - \frac{1}{x+1}}^{\varphi(x)} = x$

$x_{k+1} = x_k^2 + x_k - \frac{1}{x_k+1}$

$x_0 = 0$

$x_1 = -1$

$x_2 = 0 - \frac{1}{0}$ $\times$

||

$x_0 = 1$

$x_1 = \frac{3}{2} = 1.5$

$x_2 = \frac{67}{20} > 3$

• 1

$\varphi'(x) = 2x + 1 + \frac{1}{(x+1)^2}$

$|\varphi'(0)| = 2 > 1$

$|\varphi'(1)| = 3 + \frac{1}{4} > 1$

asi div.

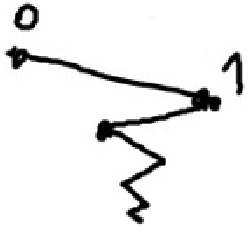
- $x^2 = \dfrac{1}{x+1} \iff \underbrace{x = \dfrac{1}{\sqrt{x+1}}}$ $\qquad \varphi(x) = \dfrac{1}{\sqrt{x+1}}$

$$x_{k+1} = \frac{1}{\sqrt{x_k+1}}$$

$(x+1)^{-1/2}$

$$\varphi'(x) = \frac{-1}{2(x+1)^{3/2}} = \frac{-1}{2\sqrt{x+1}^{\,3}}$$

$$\begin{cases} x_0 = 0 \\ x_1 = 1 \\ x_2 = \dfrac{1}{\sqrt{2}} \end{cases}$$



$|\varphi'(0)| = \left|\dfrac{-1}{2}\right| < 1$ O.K.

$|\varphi'(1)| = \dfrac{1}{2\sqrt{2}^{\,3}} < 1$

vypadá to na konv.

$x_0 \to \infty \quad |\varphi'| \to 0$

$\Rightarrow |\varphi'| \le \frac{1}{2}$ na $(0, \infty)$.

$$\underbrace{\varphi : I \longrightarrow I \quad \text{na } I : \ |\varphi'| \le q < 1}$$

$$\varphi(x) = \frac{1}{\sqrt{x+1}} \qquad x_k > 0 \Rightarrow x_{k+1} > 0$$

$$\Downarrow$$

$$x_{k+1} \le 1$$

$$\varphi : (0, 1\rangle \longrightarrow (0, 1\rangle$$

$$\text{na } \langle 0, 1\rangle : \ \underbrace{|\varphi'| \le \frac{1}{2} \quad < 1}$$

Určitě bude konvergence

- $x^2 = \dfrac{1}{x+1} \iff x+1 = \dfrac{1}{x^2} \iff x = \dfrac{1}{x^2} - 1$

$$\varphi(x) = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}$$

$$\varphi'(x) = \frac{-2}{x^3}$$

$$|\varphi'(1)| = 2 > 1$$

$$|\varphi'(\tfrac{1}{2})| = 16 > 1$$

téměř jistě divergence

$$x_{k+1} = \frac{1 - x_k^2}{x_k^2}$$

$x_0 = 0$	$x_0 = 1$	$x_0 = \frac{1}{2}$
	$x_1 = 0$	$x_1 = 3$
		$x_2 = -\frac{8}{9}$

• $$\underbrace{x^2 = \frac{1}{x+1}}_{} \iff \left[x = \underbrace{\frac{1}{x^2+x}}_{\varphi(x)} \right] = (x^2+x)^{-1}$$

$$x_{k+1} = \frac{1}{x_k^2 + x_k}$$

$x_0 = 1$

$x_1 = \frac{1}{2}$

$x_2 = \frac{4}{3}$

$$\varphi'(x) = \frac{-(2x+1)}{(x^2+x)^2}$$

$$|\varphi'(1)| = \frac{3}{4} < 1$$

$$\left|\varphi'\left(\tfrac{1}{2}\right)\right| = \frac{32}{9} > 1$$

asi div.

• $x^2 = \frac{1}{x+1} \iff x = \frac{1}{x^2} - 1 = \underbrace{\frac{1-x^2}{x}}_{\varphi(x)}$

$$\underline{\varphi_\lambda(x) = \lambda \cdot \frac{1-x^2}{x^2}} + \underline{(1-\lambda)\cdot x} \qquad x_{k+1} = x_{k}$$

$$\boxed{\lambda = \frac{1}{4}} \qquad \varphi_{\frac{1}{4}}(x) = \frac{1}{4}\cdot\frac{1-x^2}{x^2} + \frac{3}{4}\cdot x$$

$x_0 = 1$

$x_1 = \frac{3}{4}$

$x_2 = \frac{109}{144}$

$$\varphi'_{\frac{1}{4}}(x) = \frac{1}{4}\cdot\frac{-2}{x^3} + \frac{3}{4} = \frac{3x^3-2}{4x^3}$$

$$\left.\begin{aligned} \left|\varphi'_{1/4}(1)\right| &= \frac{1}{4} < 1 \\ \left|\varphi'_{1/4}\left(\tfrac{1}{2}\right)\right| &= \left|\frac{3-16}{4}\right| = \left|-\frac{13}{4}\right| > 1 \end{aligned}\right\} \text{nevím.}$$

$$\varphi_\lambda(x) = \lambda \frac{1-x^2}{x^2} + (1-\lambda)\cdot x$$

chci λ aby $|\varphi_\lambda'(x_0)| = 0$

kde?
v x_0?

$$\varphi_\lambda'(x) = \lambda \cdot \frac{-2}{x^3} + (1-\lambda)$$

$$\leftarrow \quad -1 \leftrightarrow \quad 0 \Leftrightarrow 1 \rightarrow$$

chci: $|\varphi_\lambda'(1)| = 0$

$$-2\lambda + (1-\lambda) = 0 \Longrightarrow \lambda = 1/3$$

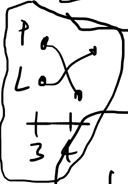
chci: $|\varphi_\lambda'(\tfrac{1}{2})| = 0$: $\lambda \cdot (-16) + 1 - \lambda = 0$

$$\boxed{\lambda = \frac{1}{17}}$$

$\boxed{x-3=\frac{1}{x^2}}$ řešení

$\underbrace{x-3-\frac{1}{x^2}}_{f(x)}=0$

	L		P
$x=3$	0	$<$	$\frac{1}{9}$
$x=4$	1	$>$	$\frac{1}{16}$



→ převod na pevný bod

→ iterační vzorec, 2 iter. kroky $\boxed{x_0=1}$

→ odhad šance na úspěch pomocí $\underline{\varphi'(1)}$ $\left(\varphi'(3)\right)$

$2x$

- $x = x-3-\frac{1}{x^2}+x = \underbrace{2x-3-\frac{1}{x^2}}_{\varphi(x)}$

$$x_{k+1}=2x_k-3-\frac{1}{x_k^2} \implies x_0=1$$

$x_1=-2$

$x_2=-7-\frac{1}{4}=-\frac{29}{4}$

$$\varphi'(x)=2+\frac{2}{x^3}$$

$$\boxed{|\varphi'(1)|=4>1}$$

$\varphi'(3),\varphi'(4)\geq 2$

$\boxed{\text{nejspíše diverguje}}$

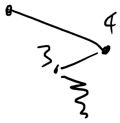
- $x-3=\frac{1}{x^2} \iff x=3+\frac{1}{x^2}$ $\qquad \varphi(x)=3+\frac{1}{x^2}$

$x_{k+1}=3+\frac{1}{x_k^2}$ $\qquad \varphi'(x)=-\frac{2}{x^3}$

$x_0=1$

$x_1=4$

$x_2=3+\frac{1}{16}=\frac{49}{16}$



$|\varphi'(1)|=2>1$

nevypadá to dobře

$x\geq 3 \Rightarrow |\varphi'|\leq\frac{2}{9}<1$

- $x - 3 = \frac{1}{x^2} \iff x^3 - 3x^2 = 1 \iff x = \sqrt{\frac{x^3 - 1}{3}} = (\ )^{1/2}$ ''

$x_{k+1} = \sqrt{\frac{x_k^3 - 1}{3}}$ $\qquad \varphi'(x) = \frac{1}{2} \frac{\frac{1}{3} \cdot 3x^2}{\sqrt{\frac{x^3-1}{3}}} = \frac{x^2}{2\sqrt{\frac{x^3-1}{3}}}$

$x_0 = 1$

$x_1 = 0$ $\qquad \varphi'(1) = \frac{1}{0^+} = \infty$ to vypadá špatně

$x_2 = \sqrt{-\frac{1}{3}}$ $\qquad \left(\left(\left(\varphi'(3) \approx \frac{9}{2 \cdot 3} \simeq 3/2\right)\right)\right)$

- $x^4 - 3x^3 = x \iff x_{k+1} = x_k^4 - 3x_k^3$ $\qquad \varphi(x) = x^4 - 3x^3$

$\varphi'(x) = 4x^3 - 9x^2$

$x_0 = \underline{1}$

$x_1 = \underline{-2}$ $\qquad |\varphi'(1)| = 5 > 1$ nejspíše diverguje

$x_2 = 16 + 24 = \underline{40}$

• $x-3=\frac{1}{x^2} \iff x^3-3x^2=1 \iff x(x^2-3x)=1$

$$\boxed{x=\frac{1}{x^2-3x}} \quad \varphi(x)$$

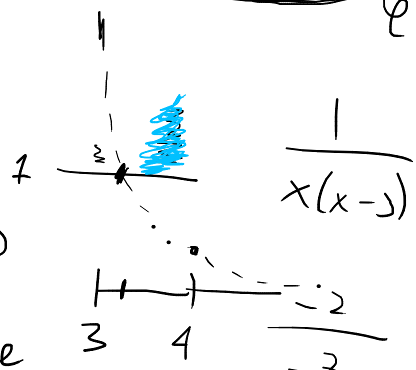
$x_{k+1}=\dots$

$x_0=1$

$x_1=-\frac{1}{2}$

$x_2=\frac{4}{7}$

$$\varphi'(x)=\frac{-(2x-3)}{(x^2-3x)^2}$$

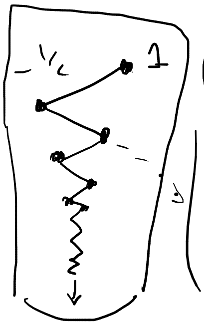


$$\frac{1}{x(x-3)}$$

$$-2 \quad \frac{-2}{-7/2} \quad (x^2-3x)^{-1}$$

$$\left|\varphi'(3)\right|=\left|\frac{1}{0}\right|=\infty$$

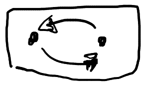
nejspíše divergence



3 4

|—|

$$\left|\varphi'(4)\right|=\left|\frac{-5}{16}\right|=\frac{5}{16}<1$$



$$\left\{ \left|\varphi'(3{,}5)\right|= \right\}$$

- $x-3=\frac{1}{x^2}$ $\quad x^2=\frac{1}{x-3}$ $\quad x=\sqrt{\frac{1}{x-3}}$

$\Downarrow$

$x^2-3x=\frac{1}{x}$

$\frac{1}{3}\left(x^2-\frac{1}{x}\right)=x$

• $x-3=\frac{1}{x^2}$

Bonus:

→ relaxační verze $\varphi_\lambda(x)$

→ iterační vzorec pro $\lambda=\frac{2}{3}$

$x_0=1$

$x_1=1$

Bonus^2

→ najdi optimální λ_{opt}

kde? $x_0=1$

$$\left.\begin{array}{l} x_0=3 \\ x_0=4 \end{array}\right\}$$

- $\varphi(x) = 2x - 3 - \frac{1}{x^2}$

$$\varphi_\lambda(x) = \lambda\cdot\left(2x - 3 - \frac{1}{x^2}\right) + (1-\lambda)\cdot x$$

$$\lambda = 2/3$$

$$x_{k+1} = \underbrace{\frac{2}{3}}\left(2x_k - 3 - \frac{1}{x_k^2}\right) + \underbrace{\frac{1}{3}}\cdot x_k$$

$$-\frac{x^3+2}{x^3}$$

$$x_0 = 1 \quad \Rightarrow \quad \underline{\underline{x_1}} = -\frac{4}{3} + \frac{1}{3} = \underline{\underline{-1}}$$

opt: $\varphi'_\lambda(x) = \lambda\cdot\left(2 + \frac{2}{x^3}\right) + (1-\lambda) \overset{?}{=} 0 \iff 1 = \lambda\left(1 - 2 - \frac{2}{x^3}\right)$

$$\lambda_{opt} = -\frac{x^3}{x^3+2}$$

$\lambda_{opt} = \underline{\underline{-\frac{27}{29}}}$, $x_0 = 3$

$\lambda_{opt} = -\frac{64}{66} = \underline{\underline{-\frac{32}{33}}}$, $x_0 = 4$

- $\varphi(x) = 3 + \frac{1}{x^2}$

$\lambda = 2/3$

$$\varphi_\lambda(x) = \lambda\left(3 + \frac{1}{x^2}\right) + (1-\lambda)\,x$$

$$x_{k+1} = \frac{2}{3}\left(3 + \frac{1}{x_k^2}\right) + \frac{1}{3}\cdot x_k$$

$x_0 = 1$, $\underline{x_1} = \frac{8}{3} + \frac{1}{3} = \underline{3}$

opt: $\varphi'_\lambda(x) = \lambda \cdot \frac{-2}{x^3} + (1-\lambda) \stackrel{!}{=} 0$

$\Leftrightarrow 1 = \lambda\left(1 + \frac{2}{x^3}\right) \Rightarrow \lambda_{opt} = \frac{x^3}{x^3+2}$

$\underline{x_0 = 3}$ $\lambda_{opt} = \frac{27}{29}$

$x_0 = 1$: $\lambda_{opt} = \frac{1}{3}$

$\underline{x_0 = 4}$ $\lambda_{opt} = \frac{32}{33}$

- $$\underbrace{\varphi(x) = \frac{1}{x^2 - 3x}}$$

$$x_{k+1} = \frac{2}{3} \cdot \frac{1}{x_k^2 - 3x_k} + \frac{1}{3} x_k$$

$$\underbrace{x_0 = 0} : \quad \underbrace{x_1 = -\frac{1}{3}}$$

opt: $$\lambda \cdot \frac{-(2x-3)}{(x^2-3x)^2} + 1 - \lambda \overset{?}{=} 0 \qquad \lambda = \frac{1}{1 + \frac{2x-3}{(x^2-3x)^2}}$$

$$x_0 = 3 \Rightarrow \frac{1}{1 + \frac{*}{0}} \quad \boxed{\lambda_{opt} = 0} \quad = \frac{(x^2-3x)^2}{(x^2-3x)^2 + 2x - 3}$$

nejde