Úloha: $\begin{bmatrix} y_1' = 2y_1 - y_2 \\ y_2' = 4y_1 - 3y_2 \end{bmatrix}$ $\begin{bmatrix} y_1(0) = 0 \\ y_2(0) = -3 \end{bmatrix}$

1) obecné řešení

- eliminace, z (#1) $y_2 = 2y_1 - y_1'$ ⊛

do (#2): $2y_1' - y_1'' = 4y_1 - 3(2y_1 - y_1') \iff$

$\iff \underbrace{y_1'' + y_1' - 2y_1 = 0}$ $\underbrace{\lambda^2 + \lambda - 2} = \underbrace{(\lambda+2)(\lambda-1)} = 0$

$\lambda = 1, -2$

$$\boxed{\begin{aligned} y_1(x) &= ae^{x} + be^{-2x} \\ y_2(x) &= ae^{x} + 4be^{-2x}, \quad x \in \mathbb{R} \end{aligned}}$$

⊛ $y_2 = 2ae^{x} + 2be^{-2x}$
$\quad - ae^{x} + 2be^{-2x}$

• <u>matice</u> $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$  vl. č.: $\det \begin{pmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{pmatrix} =$

$(\lambda-2)\cdot(\lambda+3) + 4 = \lambda^2 + \lambda - 2 = 0$

$\Rightarrow \quad \lambda = 1, -2$

$\lambda = 1$: $\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 - v_2 = 0$ zvolím: $v_2 = 1 \Rightarrow v_1 = 1$ $\vec{v_a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{y_a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{x}$

$\lambda = -2$ $\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \longrightarrow 4v_1 - v_2 = 0$ zvolíme: $v_1 = 1 \Rightarrow v_2 = 4$

$\vec{y_b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}$

obecné řeš: $\vec{y} = a \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x} + b \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}$

$= \begin{pmatrix} ae^{x} + be^{-2x} \\ ae^{x} + 4be^{-2x} \end{pmatrix}$

$\boxed{\begin{aligned} y_1(x) &= ae^{x} + be^{-2x}, \\ y_2(x) &= ae^{x} + 4be^{-2x}, \; x \in \mathbb{R} \end{aligned}}$

2) poč. podm.: chci $\begin{bmatrix} y_1(0) = a\cdot 1 + b\cdot 1 = 0 \\ y_2(0) = a\cdot 1 + 4b\cdot 1 = -3 \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} a + b = 0 \\ a + 4b = -3 \end{bmatrix} \Rightarrow \underline{(\#2) - (\#1)}: \; 3b = -3, \quad b = -1, \quad a = 1$

$$\boxed{\begin{bmatrix} y_1(x) = e^{x} - e^{-2x} \\ y_2(x) = e^{x} - 4e^{-2x} \end{bmatrix}, \; x \in \mathbb{R}}$$

✓

Bonus:

$$\boxed{\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 4y_1 - 3y_2 \end{aligned}}$$

stac. řeš.: $\left.\begin{aligned} y_1(x) &= y_{01} \\ y_2(x) &= y_{02} \end{aligned}\right\}$ $y_1' = y_2' = 0 \quad \forall x$

$\vec{y}(x) = \vec{y_0} \in \mathbb{R}$

chci $\begin{bmatrix} 2y_1 - y_2 = 0 \\ 4y_1 - 3y_2 = 0 \end{bmatrix}$

triviální řeš.:

$$\begin{pmatrix} y_1 = 0 \\ y_2 = 0 \end{pmatrix}$$

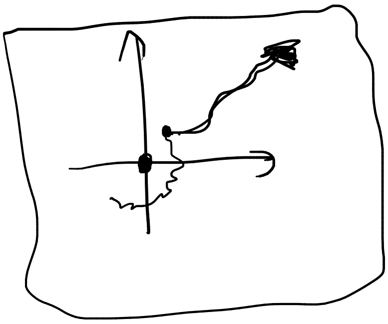
→ stac. řeš.: $\boxed{\begin{aligned} y_1(x) &= 0 \\ y_2(x) &= 0, \ x \in \mathbb{R} \end{aligned}}$

(0,0) trajektorie

(0,0) ekvilibrium

## Stabilita

$\rightsquigarrow$ $\left[\begin{array}{l} y_1(x) = ae^{x} + be^{-2x} \\ y_2(x) = ae^{x} + 4be^{-2x} \end{array}\right]$ $a \neq 0$ nebo $b \neq 0$



Co když $a \neq 0$?

pak $y_1 \to \infty$

$y_2 \to \infty$

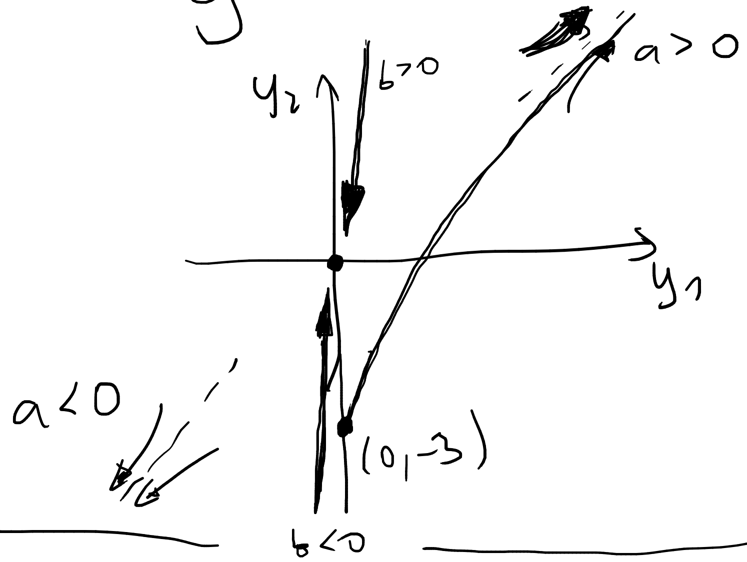
$\boxed{\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -2 \end{array}}$ $e^{x} \to \infty$

$(0,0)$ je nestabilní ekvilibr (rovnovážný bod)

$y_1(x) = y_2(x) = 0$ je nestabilní stac. řeš.

$$\left[\vec{y}(x) = a\cdot\begin{pmatrix}1\\1\end{pmatrix}e^{x} + b\begin{pmatrix}1\\4\end{pmatrix}e^{-2x}\right]$$

$$x \sim \infty : \quad \vec{y}(x) \sim a\begin{pmatrix}1\\1\end{pmatrix}e^{x} = \begin{pmatrix}1\\1\end{pmatrix}\overset{\text{číslo}}{[ae^{x}]} \to \infty$$



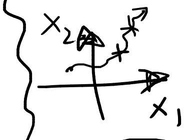

$a = 0$ ?

$$\vec{y}(x) = \begin{pmatrix}1\\4\end{pmatrix}\cdot \underbrace{be^{-2x}}_{\to\, 0}$$

$$\left[\begin{array}{l}\dot{x}_1 = x_1 + x_2\\ \dot{x}_2 = -2x_1 + 3x_2\end{array}\right]$$

obecné řeš.

~~$x_1, x_2$~~




$\vec{x}(t) = \ldots$

$$\rightsquigarrow \quad A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} = (\lambda-1)\cdot(\lambda-3)+2 = \lambda^2 - 4\lambda + 5 = 0$$

$$\underbrace{\lambda_{1,2}} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \sqrt{-1} = \boxed{2 \pm i}$$

$$\underbrace{\lambda^2 - 4\lambda + 4} + 1$$
$$(\lambda-2)^2 + 1 = 0$$

$$\boxed{\lambda = 2+i} \quad \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \longrightarrow -(1+i)v_1 + v_2 = 0$$

zvolím $v_1 = 1 \Rightarrow v_2 = 1+i$

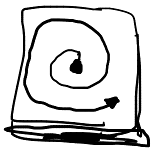
$$\vec{x}_c = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t} = e^{2t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{it} = e^{2t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\left[\cos(t) + i\sin(t)\right]$$

$$\underline{\underline{\vec{x}_c}} = e^{2t}\begin{pmatrix}1\\ 1+i\end{pmatrix}\left[\cos(t)+i\sin(t)\right] = e^{2t}\begin{pmatrix}\cos(t)+i\sin(t)\\ \cos(t)+i\sin(t)+i\cos(t)-\sin(t)\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}\cos(t)\\ \cos(t)-\sin(t)\end{pmatrix}e^{2t}}_{Re \to \vec{x}_a} + i\underbrace{\begin{pmatrix}\sin(t)\\ \cos(t)+\sin(t)\end{pmatrix}e^{2t}}_{Im \to \vec{x}_b} \qquad (\{\vec{u}, \vec{v}\}) \quad \cancel{\{\vec{u}, \vec{v}\}}$$

$Re(\vec{x}_c)$

$$\vec{x}_a = \begin{pmatrix}\cos(t)\\ \cos(t)-\sin(t)\end{pmatrix}\underline{\underline{e^{2t}}}, \qquad \vec{x}_b = \begin{pmatrix}\sin(t)\\ \cos(t)+\sin(t)\end{pmatrix}\underline{\underline{e^{2t}}}$$



a. b.

obecné řešení

$$\underline{\left[\begin{array}{l} x_1(t) = a\cos(t)\,e^{2t} + b\cdot\sin(t)\,e^{2t} \\ x_2(t) = a(\cos(t)-\sin(t))\,e^{2t} + b(\cos(t)+\sin(t))\,e^{2t},\ t\in\mathbb{R}\end{array}\right.}$$

# Transformace

$$y''' + 3y'' - 2y' - y = 0 \qquad \ast$$

$$y_1 = y \rightsquigarrow y_1''' + 3y_1'' - 2y_1' - y_1 = 0$$

$$y_1' = y_2 \rightsquigarrow y_2'' + 3y_2' - 2y_2 - y_1 = 0$$

$$y_2' = y_3 \rightsquigarrow y_3' + 3y_3 - 2y_2 - y_1 = 0$$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 + 2y_2 - 3y_3 \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix} \ast$$

$$\underbrace{y'''' + 13y''' - 23y'' + 14y' - 13y = 0}$$

$(y = y_1)$

$$y_2 = y_1' \longrightarrow y_2''' + 13y_2'' - 23y_2' + 14y_2 - 13y_1 = 0$$

$$y_3 = y_2' \longrightarrow y_3'' + 13y_3' - 23y_3 + 14y_2 - 13y_1 = 0$$

$$y_4 = y_3' \longrightarrow y_4' + 13y_4 - 23y_3 + 14y_2 - 13y_1 = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = 13y_1 - 14y_2 + 23y_3 - 13y_4 \end{cases}$$

$$\left[ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 13 & -14 & 23 & -13 \end{pmatrix} \right]$$

$$y''' + 3y'' - 2y' - y = 0$$

$$y_1 = y'$$

$$\begin{cases} y' = y_1 \\ y_1' = y_2 \\ y_2' = \underline{y} + \underline{2y_1} \underline{-3y_2} \end{cases}$$

$$\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$