Chapter 1

Sets and Relations

1.1 Sets

In this section we first recall some well known facts about sets and operations with them. You probably know them already. The main focus will be on the notion of *cardinality* of a set, which for finite sets coincides with the number of elements. The cardinality will be defined in such a way that it can be used also for infinite sets, and we show that there are infinite sets with "smaller number of elements", i.e. a smaller cardinality than other infinite sets.

Let us start with the notion of a set. A set is what mathematicians call a collection of objects that can be distinguished from each other and these objects we call elements. For a well known sets we use common notation — $\mathbb{N}$ for the set of all natural numbers, $\mathbb{Z}$ for the set of all integers, $\mathbb{Q}$ for the set of all rational numbers, and $\mathbb{R}$ for the set of all real numbers.

1.1.1 Principle of Extensionality. Two sets S and T are equal (we write $S = T$) if and only if every element of the set S is an element of the set T and conversely, every element of T is an element of S.

1.1.2 A set can be given by listing its elements, or by a property that characterizes its elements. If $p(x)$ is a property which an element has or does not have then the set C consisting of all elements having the property $p(x)$ (and no others) is denoted by

$$C = \{x \mid p(x)\}.$$

For example, the set of all squares of numbers $1, \ldots, 4$ is either $S = \{1, 4, 9, 16\}$, or as

$$S = \{x \mid x = y^2, y \in \mathbb{N}, 0 < y < 5\}.$$

The set of all even natural numbers is $\{m \mid m = 2k, k \in \mathbb{N}\}$. The set of all odd numbers is $\{m \mid m = 2k + 1, k \in \mathbb{N}\}$.

1.1.3 Subsets.
Definition. Given two sets S and T. If every element of the set S is also an element of T, we say that S is a *subset* of T, and we write $S \subseteq T$.

If $S \subseteq T$ and S and T are different sets, we also say that S is a *proper subset* of T. □

Note that equivalently we have: T is *not* a subset of S if and only if there is an element x for which $x \in T$ and $x \notin S$.

1.1.4 Proposition. For all sets S, T we have $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. □

Justification. $S \subseteq T$ and $T \subseteq S$ mean precisely that S and T have the same elements, so they are equal.

1.1.5 Empty Set. A very important role among sets is played by so called empty set.

Definition. The *empty set* is a set that contains no element; we denote it $\emptyset$. $\square$

Fact. We have $\emptyset \subseteq A$ for every set A.

Justification. It cannot happen that $\emptyset$ **is not** a subset of a set A. Indeed, $\emptyset \not\subseteq A$ means that there is $x \in \emptyset$ and $x \notin A$. But there is no $x \in \emptyset$. So, $\emptyset \not\subseteq A$ cannot be true.

1.1.6 Set Operations. We recall the well known set operations.

Given two sets A and B. Their *union* is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\};$$

their *intersection* is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *difference* of two sets A and B (in this order) is the set

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

If there is a fixed "universal" set U and $A \subseteq U$ then by the *complement* of the set A in U we mean the set $U \setminus A$. If the universe U is generally understood, then we write only $\overline{A}$ for the complement of A in U.

The *Cartesian product* of A and B (we denote it by $A \times B$) is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

If $A = B$ we speak about a *Cartesian power* of the set A and write A^2 instead of $A \times A$, A^3 is the set of all triples of elements of A, More precisely,

$$A^3 = \{(a, b, c) \,|\, a, b, c \in A\}.$$

Similarly, for natural number $n \geq 2$ we have

$$A^n = \{(a_1, a_2, \ldots, a_n) \,|\, a_i \in A\}.$$

Examples. We have $\{1, 2, 4\} \cup \{1, 4, 5, 6\} = \{1, 2, 4, 5, 6\}$ and $\{1, 2, 4\} \cap \{1, 4, 5, 6\} = \{1, 4\}$. $\{1, 2, 4\} \setminus \{1, 4, 5, 6\} = \{2\}$, $\{1, 4, 5, 6\} \setminus \{1, 2, 4\} = \{5, 6\}$.

For example, $\{1, 4\} \times \{1, 2, 6\} = \{(1, 1), (1, 2), (1, 6), (4, 1), (4, 2), (4, 6)\}$; the second Cartesian power of $A = \{1, 4\}$ is $A^2 = \{(1, 1), (1, 4), (4, 1), (4, 4)\}$.

1.1.7 Disjoint Sets.

Definition. If $A \cap B = \emptyset$, we say that the sets A and B are *disjoint*. $\square$Evidently, the set of even natural numbers is disjoint with the set of odd natural numbers, and their union is the whole set $\mathbb{N}$ of all natural numbers.

1.1.8 Power Sets.

Definition. Let A be a set. The *power set* $\mathcal{P}(A)$ of the set A is the set of all subsets of the set A; or more formally,

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$

$\square$

For example, $P(\emptyset) = \{\emptyset\}$ and $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Let us point out that a power set is always non-empty, since it contains the empty set $\emptyset$.

1.1.9 Characteristic Function of a Subset. We introduce a new notion – so called a characteristic function of a given subset. You may come across it when representing a set in computer programs. The notion can also be used for counting the number of subsets of a given set (as we see later).

Definition. A *characteristic function* χ_A of a subset $A \subseteq U$ is the mapping $\chi_A \colon U \to \{0,1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A; \\ 0 & \text{for } x \in U \setminus A. \end{cases}$$

Note that $\chi_A(x)$ can be viewed as the answer to the question: Does x belong to A? Here $\chi_A(x) = 1$ means yes, and $\chi_A(x) = 0$ means no.

Let A and B be two different subsets of the set U. Then their characteristic functions are different. Indeed, if $A \neq B$ then there is an element x which is in one of the sets and does not belong to the other set. For the sake of simplicity, let us assume that $x \in A$ and $x \notin B$. Then $\chi_A(x) = 1 \neq 0 = \chi_B(x)$ and therefore $\chi_A \neq \chi_B$.

Every mapping $\chi \colon U \to \{0,1\}$ is a characteristic function of a suitable subset of U. Indeed, define a subset C of U by $C = \{x \mid \chi(x) = 1\}$. Then $\chi = \chi_C$.

Thus subsets of a given set U and characteristic functions from U into $\{0,1\}$ are only two equivalent descriptions of the same ideal reality. A simple consequence of this fact is the following fact:

Let U be any finite set with n elements. Then there are exactly 2^n subsets of U. (In other words, for a finite set U with n elements the power set $P(U)$ has 2^n elements.)

1.2 Cardinality of Sets

In this section we present a formal approach to the intuitive notion of the size of a set number of elements), and when two sets have the same size. The reader is familiar with *finite sets.* A set is infinite if it is not a finite one. We will show that among infinite sets we can speak of sets with "the same size" and that there are infinite sets of "different sizes". At first we must recall what a bijection, i.e. a bijective mapping means.

1.2.1 Bijections. Let us recall that a mapping f from a set A into a set B is *bijective* if is injective and surjective.

- A mapping f is *injective* (or *one-to-one*) provided for every two different elements $x, y \in A$ their images $f(x)$, $f(y)$ are also different.
- A mapping is *surjective* (or *onto*) provided for every element $y \in B$ there exists an element $x \in A$ such that $y = f(x)$.

1.2.2 Sets with Same Cardinality.

Definition. Two sets A and B are said to have the *same cardinality* if there exists a bijective mapping from A onto B. This fact is denoted by $|A| = |B|$. □

Example. The set S of all even natural numbers has the same cardinality as the set T of all odd natural numbers. Indeed, the mapping

$$f \colon S \to T \text{ defined by } 2n \mapsto 2n+1$$

is a bijective mapping from S onto T.

1.2.3 Countable Sets. The "smallest" cardinality of an infinite set is the cardinality of the set of all natural numbers $\mathbb{N}$. Countable sets are in fact sets that have the same cardinality as $\mathbb{N}$.

Definition. We say that a set A is *countable* provided it has the same cardinality as the set of all natural numbers $\mathbb{N}$. If a set A is infinite and not countable then we say that it is *uncountable*. □

The following proposition helps us to easily decide whether an infinite set is countable or not.

1.2.4 Proposition. A set A is countable if and only if it can be arranged in an injective infinite sequence, (i.e. a sequence which is infinite and where no two elements are equal). □

Justification. Assume that we are able to arrange the set A in an injective infinite sequence, say

$$A = \{a_0, a_1, \dots, a_n, \dots\}.$$

Define a mapping $f\colon \mathbb{N} \to A$ by $f(i) = a_i$ for every $i \in \mathbb{N}$. Then f is a desired injective mapping from the set $\mathbb{N}$ onto the set A, i.e. a bijection.

Assume we have a bijective mapping f from $\mathbb{N}$ onto A. Then the set A can be written as follows: $A = \{f(0), f(1), \dots, f(n), \dots\}$. In other words, we have arranged the set A into an injective infinite sequence.

1.2.5 Example of a Countable Set. The set of all integers $\mathbb{Z}$ is countable.

Indeed, $\mathbb{Z}$ can be arranged into an injective infinite sequence. We do it as follows

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \dots, n, -n, \dots.$$

More precisely, the number 0 will be in the 0-th place (i.e. 0 is the element a_0), the number 1 in the first place (the element a_1), the number -1 in the second place (the element a_2), the number 2 in the place $2 \cdot 2 - 1 = 3$ (the element a_3), the number -2 in the place $2 \cdot 2 = 4$ (the element a_4), etc. Generally, a positive integer n will be in the place $2n - 1$ (the element a_{2n-1}) and the number $-n$ in the place $2n$ (the element a_{2n}).

Further, we give three propositions about countable sets. The first one in fact shows that a subset of countable set cannot be uncountable; the next two show some set operations that maintain countability.

1.2.6 Proposition. Any infinite subset of a countable set is again countable. □

Justification. If we can arrange the set A into an injective infinite sequence $\{a_0, a_1, \dots\}$ then each of its infinite subsets $B \subseteq A$ can be obtained as a subsequence of the sequence $\{a_0, a_1, \dots\}$ which is again injective and infinite.

1.2.7 Proposition. If two sets are countable so is their union. □

Justification. Let $A = \{a_0, a_1, a_2, \dots\}$ and $B = \{b_0, b_1, b_2, \dots\}$ be two countable sets. Their union $C = A \cup B$ may be written as follows

$$A \cup B = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\}.$$

More formally, $A \cup B = \{c_0, c_1, \dots\}$, where $c_{2n} = a_n$ and $c_{2n+1} = b_n$. This sequence does not need to be injective. Indeed, the elements that are in A and simultaneously in B are in the above sequence twice. But by omitting the second appearance of them we get an injective sequence. Therefore the set $A \cup B$ is countable.

1.2.8 Proposition. The Cartesian product of two sets is a countable set. □

Justification. We will show that the set $C = A \times B$, for A and B countable sets, can be arranged into an injective infinite sequence. Let $A = \{a_0, a_1, a_2, \ldots\}$ and $B = \{b_0, b_1, b_2, \ldots\}$ be injective sequences. We will make use of the following scheme.

$$\begin{array}{ccccccc}
(a_0, b_0) & & (a_0, b_1) & & (a_0, b_2) & & \ldots \\
& \swarrow & & \swarrow & & & \\
(a_1, b_0) & & (a_1, b_1) & & (a_1, b_2) & & \ldots \\
& \swarrow & & \swarrow & & & \\
(a_2, b_0) & & (a_2, b_1) & & (a_2, b_2) & & \ldots \\
& \swarrow & & \swarrow & & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The arrangement of C is seen from the scheme. Indeed,

$$C = \{(a_0, b_0), (a_0, b_1), (a_1, b_0), (a_0, b_2), (a_1, b_1), (a_2, b_0), (a_0, b_3), \ldots\}.$$

Here is the precise description of the sequence: The pair (a_i, b_j) will be in the place $k = i + \frac{(i+j)(i+j+1)}{2}$ of the sequence. Notice that in this case the constructed sequence is injective.

1.2.9 We will use the above proposition to show that well known sets are countable.

Example. The set $\mathbb{Q}$ of all rational numbers is countable.

Justification. Indeed, every rational number can be represented as a fraction $\frac{p}{q}$ where q is a non-zero natural number and p is an integer. Fractions can be viewed as ordered pairs (p, q) where p is the numerator and q the denominator of the fraction $\frac{p}{q}$. Since the set of all integers is countable as well as the set of all non-zero natural numbers, according to 1.2.8 we get that the set M of all ordered pairs (p, q) is countable. The set $\mathbb{Q}$ is now the subset of the set M which contains only those pair (p, q) for which p and q are relatively prime. Since the set $\mathbb{Q}$ is infinite, it is countable by the proposition 1.2.6.

1.2.10 Proposition. The union of a countable system of finite disjoint sets is again countable.

In other words, if $A_0, A_1, \ldots A_n, \ldots$ are finite disjoint sets then their union $A_0 \cup A_1 \cup \ldots = \bigcup_{i \in \mathbb{N}} A_i$ is countable as well. □

Remark. The above proposition can be generalized to countable union of countable sets, but the proof needs axiom of choice. This is beyond the scope of our course.

1.2.11 Uncountable Sets. Recall that an infinite set is said to be uncountable if it is not countable.

In the next paragraph we will show that the set of all subsets of natural numbers is uncountable. To prove this we will prove the following:

1.2.12 Cantor Diagonal Method. The set of all infinite sequences of 0's and 1's is uncountable. □

Justification. By contradiction. Assume that the set of all infinite sequences of zeros and ones is countable. It means that we are able to arrange it into an infinite sequence. We list all the sequences its elements in the following scheme — in the first row we have the sequence s_0, in the second row the sequence s_1, in the third one s_2, etc.

$$
\begin{array}{lcllllllll}
s_0 & = & \boxed{s_0(0),} & s_0(1), & s_0(2), & s_0(3), & s_0(4), & s_0(5), & \ldots \\
s_1 & = & s_1(0), & \boxed{s_1(1),} & s_1(2), & s_1(3), & s_1(4), & s_1(5), & \ldots \\
s_2 & = & s_2(0), & s_2(1), & \boxed{s_2(2),} & s_2(3), & s_2(4), & s_2(5), & \ldots \\
s_3 & = & s_3(0), & s_3(1), & s_3(2), & \boxed{s_3(3),} & s_3(4), & s_3(5), & \ldots \\
s_4 & = & s_4(0), & s_4(1), & s_4(2), & s_4(3), & \boxed{s_4(4),} & s_4(5), & \ldots \\
s_5 & = & s_5(0), & s_5(1), & s_5(2), & s_5(3), & s_5(4), & \boxed{s_5(5),} & \ldots \\
\vdots & & & & & & & &
\end{array}
$$

We will construct a new sequence of 0's and 1's and we will show that this new sequence **is not included** into the list above. It is the sequence $\overline{s}$ defined as follows: The sequence $\overline{s}$ begins with 0 if in the first frame above there is 1, and begins with 1 if in the first frame there is 0. In other words, $\overline{s}(0) = 1$ provided $s_0(0) = 0$ and $\overline{s}(0) = 0$ provided $s_0(0) = 1$. Further we proceed analogously. If $s_1(1) = 0$, then $\overline{s}(1) = 1$, and if $s_1(1) = 1$, then $\overline{s}(1) = 0$ (notice that in the above scheme the value $s_1(1)$ is put into a frame). The value of $\overline{s}(2)$ is the number $1 - s_2(2)$, etc.

More formally: $\overline{s} = \{\overline{s}(0), \overline{s}(1), \overline{s}(2), \ldots, \overline{s}(n), \ldots\}$, where $\overline{s}(n) = 1 - s_n(n)$ for all $n \in \mathbb{N}$.

The sequence $\overline{s}$ is not equal to any of the sequences $s_0, s_1, s_2, \ldots, s_n, \ldots$. Indeed, it differs from s_0 in the zero-th place ($\overline{s}(0) = 1 - s_0(0)$), it differs from s_1 in the first place ($\overline{s}(1) = 1 - s_1(1)$), …it differs from s_n in the n-th place ($\overline{s}(n) = 1 - s_n(n)$). This means that we have not listed *all* sequences (there is at least one missing — $\overline{s}$), a contradiction.

Therefore the set of all infinite sequences of zeros and ones is not countable.

1.2.13 Theorem. The set of all subsets of natural numbers $\mathbb{N}$ is uncountable. □

Justification. Since each characteristic function of a $A \subseteq \mathbb{N}$ is an infinite sequence of 0's and 1's, the theorem follows from the Cantor diagonal method 1.2.12.