1.3 Binary relations

In mathematics, as in everyday situations, we often speak about a relationship between objects, which means an idea of two objects being related or associated one to another in some way. The notion of a binary relation makes this precise. Let us start with some examples.

1. To be a grandfather. Objects of our consideration are people; a person a is associated with a person b if a is a grandfather of b.

2. To be of the same length. Objects of our consideration are sticks; a stick a is associated with another stick b if both sticks have the same length.

3. To be a subset. Objects of our consideration are subsets of a given set U; a subset X is related to a subset Y if X is a subset of Y.

4. To be greater or equal. Objects of our consideration are numbers; a number n is related to a number m if n is greater than or equal to m.

5. To be a student of a study group. Objects of our consideration are first year students and study groups; a student a is related to a study group number K if student a belongs to study group K.

6. The sine function. Consider real numbers; a number x is related to a number y if $y = \sin x$.

1.3.1 Definition. A *relation* (more precisely a *binary relation*) *from a set A into a set B* is any set of ordered pairs $R \subseteq A \times B$. If $A = B$ we speak about a *relation on a set A*. □

We can construct new relations from others. Since a relation is a set of ordered pairs, we can use set operations for construction of new relations. But there are also specific operations – inverse relation and composition of relations. First we start with set operations.

1.3.2 Set Operations with Relations.

Definition. We say that a relation R is a *sub relation* of a relation S if $R \subseteq S$; i.e. if $a\,R\,b$, then also $a\,S\,b$. □

Definition. Let R and S be two relations from a set A into a set B. The *intersection* of relations R and S is the relation $R \cap S$; the *union* of R and S is the relation $R \cup S$; the *complement* of R is the relation $\overline{R} = (A \times B) \setminus R$. □

For example, let T be equality on the set of all real numbers $\mathbb{R}$, and S be the relation "smaller than" on $\mathbb{R}$. Then $T \cap S = \emptyset$ and $T \cup S$ is the relation to be smaller than or equal to. The complement of the relation T is non-equality on $\mathbb{R}$; i.e. the relation $\overline{T} = \{(a,b) \mid a,b \in \mathbb{R}, a \neq b\}$.

1.3.3 Inverse Relation.

Definition. Let R be a relation from a set A into a set B. Then the *inverse relation* of the relation R is the relation R^{-1} from set B into set A, defined by:

$$x\,R^{-1}y \quad \text{if and only if} \quad y\,R\,x.$$

□

Notice that the inverse relation R^{-1} to R always exists. So if R is a function then the relation R^{-1} exists; on the other hand, R^{-1} does not need to be a function.

1.3.4 Composition of Relations.

Definition. Let R be a relation from a set A into a set B and S be a relation from the set B into a set C. Then the *composition of the relations* (sometimes also called the *product*), $R \circ S$, is the relation from the set A into the set C defined by:

$$a\,(R \circ S)\,c \quad \text{if and only if there is an element } b \in B \text{ such that } \; a\,R\,b \; \text{ and } \; b\,S\,c.$$

□

1.3.5 Properties of Composition of Relations. We will show some properties of composition of relation. First, we prove that a composition of relations is associative, then that it is not commutative. (Roughly speaking, we do not need to use parentheses, but we cannot change the order.)

Proposition. The composition of relations is associative. More precisely, if R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D then

$$R \circ (S \circ T) \,=\, (R \circ S) \circ T\,.$$

□

Justification: It is not difficult to show hat for all elements $a \in A$, $d \in D$ it holds: $a\,R\circ(S\circ T)\,d$ if and only if there exist $b \in B$, $c \in C$ such that $a\,R\,b$, $b\,S\,c$ and $c\,T\,d$. And this means that $a\,(R \circ S) \circ T\,d$.

Proposition. The composition of relations is not commutative. It is not the case that $R \circ S \,=\, S \circ R$ holds for all relations R and S. □

Justification. To show the above proposition it suffices to find two relations S and T for which $R \circ S \,=\, S \circ R$ does not hold despite of the fact that both compositions exist.

Here is an example: Let A be the set of all people in the Czech Republic. Consider the following two relations R, S defined on A:

$$\begin{aligned} a\,R\,b &\quad \text{if and only if } a \text{ is a brother or a sister of } b \text{ and } a \neq b \\ c\,S\,d &\quad \text{if and only if } c \text{ is a child of } d. \end{aligned}$$

To show that $R \circ S \neq S \circ R$ it suffices to find two people x, y such that $x\,R \circ S\,y$ holds and $x\,S \circ R\,y$ does not hold. Consider any pair of a nephew a and an uncle b. We have $a\,S \circ R\,b$ since a parent of a is a brother or a sister of the uncle b. On the other hand, $a\,R \circ S\,b$ does not hold. Indeed, it would mean that one of the brothers or sisters of a were a parent of uncle b.

1.3.6 Relations on a Set. In applications an important role play relations $S \subseteq A \times B$ where $A = B$. Recall that such relations are called *relations on A*.

1.3.7 Different Types of Relations on A. Relations on a set A may have different properties. We will be mainly interested in four of them: reflexivity, symmetry, antisymmetry and transitivity. Here are the definitions:

Definition. We say that relation R on set A is

1. *reflexive* if for every $a \in A$ we have $a\,R\,a$;
2. *symmetric* if for every $a, b \in A$ it holds that: $a\,R\,b$ implies $b\,R\,a$;
3. *antisymmetric*, if for every $a, b \in A$ it holds that: $a\,R\,b$ and $b\,R\,a$ imply $a = b$;
4. *transitive*, if for every $a, b, c \in A$ it holds that: if $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$.

□

Examples. Consider the relation of non-equality R on the set of all natural numbers $\mathbb{N}$; (i.e. $n\,R\,m$ if and only if n and m are different natural numbers). This relation is not reflexive because for no $n \in \mathbb{N}$ do we have $n \neq n$. It is symmetric: If $n \neq m$ then also $m \neq n$. Relation R is not antisymmetric because e.g. $2 \neq 3$, $3 \neq 2$, and 2 and 3 are different numbers. (That is $2\,R\,3$ and $3\,R\,2$ and at the same time $2 \neq 3$.) This relation is not transitive because for example we have $2 \neq 3$ and $3 \neq 2$, and at the same time $2 = 2$ (i.e. $2\,R\,3$ and $3\,R\,2$ and it is not true $2\,R\,2$).

Relation "to be smaller than or equal to" $\leq$ on set $\mathbb{R}$ is reflexive, since $a \leq a$ for every a. It is also antisymmetric, since whenever for two numbers a, b we have $a \leq b$ and $b \leq a$, then $a = b$. It is also transitive, since if $a \leq b$ and $b \leq c$, then also $a \leq c$.

1.3.8 Equivalence Relations. One of the most important type of relations on A is so called equivalence relation. Let us recall the tautological equivalence of propositional formulas. It is one example of equivalence relation on the set of all propositional formulas. Have in mind that an "equivalence relation" on A is some sort of "generalized equality" of elements of A.

Definition. A relation R on a set A is called an *equivalence*, if it is reflexive, symmetric and transitive. □

Example. The following relation R on the set of all integers $\mathbb{Z}$, defined by:

$$m\,R\,n \quad \text{if and only if} \quad m - n \text{ is divisible by } 12, \ (m, n \in \mathbb{Z}),$$

is an equivalence relation.

Justification. Relation R is reflexive. Indeed, for every $m \in \mathbb{Z}$ we have $m - m = 0$, and zero is divisible by 12. Hence $m\,R\,m$.

Relation R is also symmetric. Indeed, if $m\,R\,n$, i.e., $m - n = 12k$ for some k, then also $n - m$ is divisible by 12 ($n - m = -12k$). Therefore $n\,R\,m$.

Moreover, R is transitive: Take any numbers $m, n, p \in \mathbb{Z}$ such that $m\,R\,n$ and $n\,R\,p$. This means $m - n = 12k$ and $n - p = 12l$ for some k and l. Then $m - p = (m - n) + (n - p) = 12k + 12l = 12(k + l)$. Hence we have $m\,R\,p$.

1.3.9 Equivalence Classes. Every equivalence relation R on A "divides" A into the sets of equivalent elements. These sets are called equivalence classes. We will see the importance of equivalence classes later when we introduce so called residue classes.

Definition. Let R be an equivalence relation on a set A. An *equivalence class* of R corresponding to the element $a \in A$ is the set $R\,[a] = \{b \in A \mid a\,R\,b\}$. □

Example: Given the equivalence relation from 1.3.8. There are 12 different equivalence classes of R; namely

$$R\,[i] = \{j \mid j = i + 12k, k \in \mathbb{Z}\}, \ \text{ for } i = 0, \ldots, 11.$$

Definition. Let R be an equivalence relation on A. The set

$$\{R[a] \mid a \in A\}$$

is called the *quotient set* and denoted by A/R.

1.3.10 Property of the Set of Equivalence Classes. The next proposition gives properties that sets of equivalence classes have.

Proposition. Let R be an equivalence relation on a set A. The set $\{\,R\,[a] \mid a \in A\}$ has the following properties:

1. Every element $a \in A$ belongs to $R\,[a]$ and hence $\bigcup\{\,R\,[a] \mid a \in A\} = A$.

2. Equivalence classes $R[a]$ are pairwise disjoint. That is, if $R[a] \cap R[b] \neq \emptyset$, then $R[a] = R[b]$.

$\square$

Justification. Since every element $a \in A$ is related to itself (reflexivity), we get $a \in R[a]$. Thus $A \subseteq \bigcup\{ R[a] \mid a \in A\}$. Moreover, each equivalence class is a subset of A, and therefore $\bigcup\{ R[a] \mid a \in A\} \subseteq A$. We have shown the first property.

Let us verify the second property. Assume that there are two classes with non-empty intersection. We will show that they coincide. Take an element $z \in R[a] \cap R[b]$. Then $a\,R\,z$ and $b\,R\,z$. Since R is symmetric, we have $z\,R\,b$. Furthermore, since $a\,R\,z$ and $z\,R\,b$, it follows from transitivity of R that $a\,R\,b$. We have shown: If two classes $R[a]$, $R[b]$ have non-empty intersection, then the elements a and b are equivalent. Now, take any element $c \in R[a]$. Then $c\,R\,a$. From transitivity of R and from $a\,R\,b$ we get that $c\,R\,b$. Hence $c \in R[b]$. Analogously one can show that every element $c \in R[b]$ also belongs to $R[a]$. Therefore $R[a] = R[b]$.