1.3.11 Partial Order, a Poset. Apart from equivalence relations there is another type of relations that plays a special role in mathematics. And it is a so called partial order, or a partial ordered set, shortly a poset.

Definition. A relation R on a set A is called an *order (partial order)*, if it is reflexive, antisymmetric and transitive. A set A together with a partial order is often called a *poset.* □

1.3.12 Examples of Posets.

1. The well-known ordering of real numbers is an order in the above sense. Indeed, for all real numbers $a, b, c \in \mathbb{R}$ we have: $a \leq a$; if $a \leq b$ and $b \leq a$ then necessarily $a = b$; if $a \leq b$ and $b \leq c$ then also $a \leq c$.

2. Denote by A the set of all subsets of the set U. Then the relation $\subseteq$, "to be a subset", is an order on A. Verification of reflexivity, antisymmetry and transitivity is left to the reader.

3. Let $A = \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. The the relation of divisibility defined by $m \,|\, n$ if and only if m is a divisor of n (i.e. if $n = k \cdot m$ for some $k \in \mathbb{N}$) is an order. Indeed, for all natural numbers m, n, k we have $m \,|\, m$; if $m \,|\, n$ and $n \,|\, m$ then $m = n$; if $m \,|\, n$ and $n \,|\, k$ then also $m \,|\, k$.

1.3.13 Proposition. If $\sqsubseteq$ is a partial order on a set A, then so is a restriction of $\sqsubseteq$ on any subset $B \subseteq A$.

1.3.14 Hasse diagram of a poset. Let $(A, \sqsubseteq)$ is a poset for a finite set A. The *covering relation* $\prec$ is a subrelation of $\sqsubseteq$ defined by

$$a \prec b \text{ if and only if } a \neq b \text{ and if } a \sqsubseteq c \sqsubseteq b \text{ then } a = c \text{ or } b = c.$$

The *Hasse diagram* contains points for all $a \in A$, representing the covering relation where elements smaller are drawn lower than the bigger ones.

1.3.15 Smallest (or least) element, greatest (or biggest) element. Given a poset $(A, \sqsubseteq)$.

- We say that $a \in A$ is the *smallest element* if for every $b \in A$ we have $a \sqsubseteq b$.
- We say that $a \in A$ is the *greatest element* if for every $b \in A$ we have $b \sqsubseteq a$.

1.3.16 Minimal elements, maximal elements. Given a poset $(A, \sqsubseteq)$.

- We say that $a \in A$ is a *minimal element* if for every $b \in A$ we have if $b \sqsubseteq a$ then $b = a$.
- We say that $a \in A$ is a *maximal element* if for every $b \in A$ we have if $a \sqsubseteq b$ then $b = a$.

1.3.17 Facts. Given a poset $(A, \sqsubseteq)$.

- If it has the smallest element, then it is the only minimal element.
- If it has the greatest element, then it is the only maximal element.
- A poset can have more than one minimal and/or maximal element, but i this case it does not have the smallest and/or greatest element.
- A poset can have no minimal and/or no maximal element.
- Any post $(A, \sqsubseteq)$ with a finite set A has at least one minimal and at least one maximal element.

1.3.18 Linear order, comparable and incomparable elements. Let $(A, \sqsubseteq)$ be a poset and $a, b \in A$. We say that a, b are *comparable* if $a \sqsubseteq b$ or $b \sqsubseteq a$. Otherwise, they are called *incomparable.*

A partial order $\sqsubseteq$ on A is called a *linear order* if any two elements of A are comparable.

1.3.19 Well-ordering. A partial order $\sqsubseteq$ on A is called *well-ordering* if any non-empty subset $M \subseteq$ has the smallest element.

Note, that a well-ordering is necessary a linear ordering; indeed, that any $\{a, b\} \subseteq A$ then if the smallest element is a then $a \sqsubseteq b$, if it is b then $b \sqsubseteq a$.

1.3.20 Well-ordering Principle. Let $\mathbb{N}$ be the set of all natural numbers. Then the ordinary relation $\leq$ "to be smaller or equal to" is a well-ordering.

Remark. Well-ordering Principle cannot be either proved or disproved. We show later that it is equivalent with the Principle of Mathematical Induction.

1.4 Mathematical Induction.

Mathematical induction is not only a device for proving assertions, but it can serve as a tool for finding formulas and for defining sets. Let us start with a formulation of a weak principle of mathematical induction.

1.4.1 Principle of (Weak) Mathematical Induction. Given a property $V(n)$ that may be true or false for $n \in \mathbb{N}$. Let n_0 be a natural number. Assume that the following two conditions hold:

1) $V(n_0)$ holds.

2w) If $V(n)$ is true for a natural number $n \geq n_0$ then so is $V(n+1)$.

Then $V(n)$ is true for all natural numbers $n \geq n_0$.

The condition 1) is called the *basic step* and the condition 2w) the *inductive step.* Moreover, the assumption that $V(n)$ is true is called the *induction assumption* (or *inductive hypothesis*).

1.4.2 Example. Let us prove, using the mathematical induction, that for any set U with n elements, it holds that the set $\mathcal{P}(U)$ of subsets of U has 2^n elements for any natural number $n \geq 0$.

Solution. We shall proceed by mathematical induction. Denote $U = \{x_1, x_2, \dots, x_n\}$.

Basic step: For $n = 0$ we have $U = \emptyset$, and $\emptyset$ has just $1 = 2^0$ subsets. Hence, the assertion holds for $n = 0$.

Inductive step: Assume that any n element set has 2^n subsets (the induction assumption).

Consider any $n+1$ element set U, i.e. $U = \{x_1, \dots, x_n, x_{n+1}\}$. We can divide subsets of U into two (disjoint) subsets A and B as follows: $A = \{X \subseteq U \mid x_{n+1} \notin X\}$ and $B = \{X \subseteq U \mid x_{n+1} \in X\}$.

By induction assumption, A has 2^n elements because it is a set of all subsets of $\{x_1, \dots, x_n\}$. Moreover,

$$B = \{Y \cup \{x_{n+1}\} \mid Y \subseteq \{x_1, \dots, x_n\}\}.$$

Therefore, B has also 2^n elements. Hence, we get

$$|\mathcal{P}(\{x_1, \dots, x_{n+1}\})| = 2^n + 2^n = 2^{n+1}$$

which proves that any $n+1$ element set U has $2n+1$ subsets.

We have proved that any set with n element has 2^n subsets for **every** $n \in \mathbb{N}$.

1.4.3 The next example shows how we can use the principle of mathematical induction to derive a formula.

Example. Derive a formula for $\sum_{i=0}^{n} i^2$.

Solution. Our guess will be that it as some polynomial of degree 3. (It is a generalization of the fact the $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.) A general polynomial of degree 3 is $a\,n^3 + b\,n^2 + c\,n + d$.

If $\sum_{i=0}^{n} i^2 = a\,n^3 + b\,n^2 + c\,n + d$ is a correct formula then it has to be valid for any $n \geq 0$. If we would like to prove the formula using mathematical induction we should have to show that the unknown coefficients a, b, c, d satisfy the basic and the inductive step.

Let us first examine the basic step:
For $n = 0$, it must hold

$$0 = \sum_{i=0}^{0} i^2 = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = d.$$

Hence $d = 0$ and if the formula would be correct then $\sum_{i=0}^{n} i^2 = a\,n^3 + b\,n^2 + c\,n$.

Now, let us deal with the inductive step. Assume that $\sum_{i=0}^{n} i^2 = a\,n^3 + b\,n^2 + c\,n$. We have to show that $\sum_{i=0}^{n+1} i^2 = a\,(n+1)^3 + b\,(n+1)^2 + c\,(n+1)$. We know that

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2.$$

Hence the coefficients a, b, c have to satisfy

$$a\,(n+1)^3 + b\,(n+1)^2 + c\,(n+1) = a\,n^3 + b\,n^2 + c\,n + (n+1)^3.$$

Two polynomials are equal if and only if the corresponding coefficients are equal. We get the following system of linear equations

$$\begin{array}{ccccccc} 3a & & & & & = & 1 \\ 3a & + & 2b & & & = & 2 \\ a & + & b & + & c & = & 1 \end{array}$$

The only solution of the system above is $a = \frac{1}{3}$, $b = \frac{1}{2}$, and $c = \frac{1}{6}$. Therefore,

$$\sum_{i=0}^{n} i^2 = \frac{1}{3}\,n^3 + \frac{1}{2}\,n^2 + \frac{1}{6}\,n = \frac{n(2n+1)(n+1)}{6}.$$

1.4.4 Theorem. The principle of mathematical induction follows from the well-ordering principle.

Proof. Assume that for a property $V(n)$ there is $n_0 \in \mathbb{N}$ such that the following conditions hold:

1. $V(n_0)$ is true.
2. If $V(n)$ is true for a natural number $n \geq n_0$ then so is $V(n+1)$.

Denote by M the set of all natural numbers $n \geq n_0$ for which $V(n)$ is not true. Assume that M has the smallest element, say n_1. Clearly, $n_1 \neq n_0$ because $V(n_0)$ is true. Hence, $n_1 > n_0$. Take $n_2 = n_1 - 1$; then $n_2 \geq n_0$. Since n_1 is the smallest element of M, $n_2 \notin M$, which means that $V(n_2)$ is true. But then by the condition 2), $V(n_1)$ it true as well, indeed, $n_1 = n_2 + 1$. A contradiction.

Therefore, M does not have the smallest element which, by well-ordering principle, means that $M = \emptyset$. We have shown that $V(n)$ is true for all natural numbers $n \geq n_0$, which is what mathematical induction claims.