Chapter 4

Difference Equations, Reccurence Equations

Difference equations, also called reccurence equations, are equations where we seek sequences that "solve" an equation of a special type. We will deal only with linear difference equations, and linear difference equations with constant coefficients into more details.

4.1 Linear Difference Equations

4.1.1 Sequences. A *sequence* is a mapping from the set of all integers greater or equal to an integer n_0 into the set of all real numbers. Hence

$$\{a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots\} \quad \text{where } a_i \in \mathbb{R}.$$

Note that in most cases $n_0 = 0$ or $n_0 = 1$, but this does not need to be always true.

4.1.2 Linear Difference Equations. Let $c_i(n)$, $i \in \{0, \ldots, k-1\}$, be functions $\mathbb{Z} \to \mathbb{R}$, $c_0(n)$ not identically zero, and let $\{b_n\}_{n=n_0}^{\infty}$ be a sequence. Then the equation

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + c_{k-2}(n)\, a_{n+k-2} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = b_n,\ n \geq n_0 \quad (4.1)$$

is called a *linear difference equation of order* k (also a *linear reccurence equation of order* k).

Functions $c_i(n)$ are called *coefficients* of the equation, the sequence $\{b_n\}_{n=n_0}^{\infty}$ the *right-hand side* of the equation.

If $\{b_n\}_{n=n_0}^{\infty}$ is the zero sequence (i.e. $b_n = 0$ for all $n \geq n_0$) then we speak about *homogeneous* equation, otherwise the equation is called *non-homogeneous*.

We can write a linear difference equation 4.1 in the following form:

$$a_{n+k} + \sum_{i=0}^{k-1} c_i(n)\, a_{n+i} = b_n,\ n \geq n_0.$$

4.1.3 Remark. In some literature, the above linear difference equation can be written

$$a_n + \sum_{i=1}^{k} c_i(n)\, a_{n-i} = b_{n-k}.$$

4.1.4 Solutions of Linear Difference Equations. A *solution* of a linear difference equation is any sequence $\{u_n\}_{n=n_0}^{\infty}$ such that if we substitute u_n for a_n in 4.1 we obtain a statement that is valid.

4.1.5 Initial Conditions. Let

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + c_{k-2}(n)\, a_{n+k-2} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = b_n,\ n \geq n_0$$

be a linear difference equation of order k. By *initial conditions* we mean the following system

$$a_{n_0} = A_0,\ a_{n_0+1} = A_1,\ \ldots,\ a_{n_0+k-1} = A_{k-1}, \tag{4.2}$$

where A_i are real numbers.

If for a solution $\{a_n\}_{n=n_0}^{\infty}$ of a difference equation 4.1 we have $a_{n_0} = A_0$, $a_{n_0+1} = A_1$, $\ldots$, $a_{n_0+k-1} = A_{k-1}$, then we say that $\{a_n\}_{n=n_0}^{\infty}$ is the *solution of 4.1 satisfying the initial conditions 4.2.*

4.1.6 The Associated Homogeneous Equation. Given a linear difference equation

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + c_{k-2}(n)\, a_{n+k-2} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = b_n,\ n \geq n_0. \tag{4.3}$$

Then the equation

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + c_{k-2}(n)\, a_{n+k-2} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = 0,\ n \geq n_0. \tag{4.4}$$

is called the *associated homogeneous equation* to the equation 4.3.

4.1.7 Proposition. Given a linear difference equation 4.3. Then the following holds:

1. If $\{u_n\}_{n=n_0}^{\infty}$ and $\{v_n\}_{n=n_0}^{\infty}$ are two solutions of the equation 4.3 then $\{u_n\}_{n=n_0}^{\infty} - \{v_n\}_{n=n_0}^{\infty}$ is a solution of the associated homogeneous equation 4.4.

2. If $\{u_n\}_{n=n_0}^{\infty}$ is a solution of 4.3 and $\{w_n\}_{n=n_0}^{\infty}$ is a solution of the associated homogeneous equation 4.4, then $\{u_n\}_{n=n_0}^{\infty} + \{w_n\}_{n=n_0}^{\infty}$ is a solution of the equation 4.3.

3. Let $\{\hat{u}_n\}_{n=n_0}^{\infty}$ be a fixed solution of the equation 4.3. Then for every solution $\{v_n\}_{n=n_0}^{\infty}$ of 4.3 there exists a solution solution $\{w_n\}_{n=n_0}^{\infty}$ of the associated homogeneous equation 4.4 for which
$$\{v_n\}_{n=n_0}^{\infty} = \{\hat{u}_n\}_{n=n_0}^{\infty} + \{w_n\}_{n=n_0}^{\infty}.$$

Indeed, we have $\{v_n\}_{n=n_0}^{\infty} = \{\hat{u}\}_{n=n_0}^{\infty} + (\{v_n\}_{n=n_0}^{\infty} - \{\hat{u}\}_{n+0}^{\infty})$.

4.1.8 Remark. According to the proposition above, to find any solution of 4.3 it suffices to find **one** solution of 4.3 and all solutions of the associated homogeneous equation 4.4. Therefore, we will be interested in solutions of homogeneous equations first.

4.1.9 Theorem. Given a homogeneous linear difference equation 4.4. Then for the set S of all solutions the following holds:

1. If $\{u_n\}_{n=n_0}^{\infty}$ and $\{v_n\}_{n=n_0}^{\infty}$ belong to S then so does $\{u_n\}_{n=n_0}^{\infty} + \{v_n\}_{n=n_0}^{\infty}$.

2. If $\{u_n\}_{n=n_0}^{\infty}$ belongs to S and α is any real number, then $\{k\, u_n\}_{n=n_0}^{\infty}$ belongs to S as well.

In other words, S is a linear space over $\mathbb{R}$, a subspace of the linear space containing all real sequences.

4.2 Linear Difference Equations with Constant Coefficients

We will concentrate on solving linear difference equations of special type – those for which the coefficients $c_i(n)$ are all constant functions. A general linear difference equation with constant coefficients is

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + c_{k-2}\, a_{n+k-2} + \ldots + c_1\, a_{n+1} + c_0\, a_n = b_n,\ n \geq n_0,\ c_i \in \mathbb{R}, \tag{4.5}$$

and its associated homogeneous equation

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + c_{k-2}\, a_{n+k-2} + \ldots + c_1\, a_{n+1} + c_0\, a_n = 0,\ n \geq n_0. \tag{4.6}$$

Note that both 4.1.7 and 4.1.9 holds also for solutions of linear difference equation with constant coefficients.

4.2.1 The Characteristic Equation. Consider equation 4.6. Assume that its solution has the form $a_n = \lambda^n$ for some real number λ. Then substitutingn it into 4.6 we get

$$\lambda^{n+k} + c_{k-1}\lambda^{n+k-1} + \ldots + c_1\lambda^{n+1} + c_0\lambda^n = 0.$$

If we cancel the equation above by λ^n we get

$$\lambda^k + c_{k-1}\lambda^{k-1} + \ldots + c_1\lambda + c_0 = 0. \tag{4.7}$$

The equation 4.7 is called the *characteristic equation of 4.6.* Any λ satisfying 4.7 leads to one solution $a_n = \{\lambda^n\}_{n=n_0}^{\infty}$.

Moreover, for distinct λ_1, λ_2 the corresponding solutions are linearly independent.

4.2.2 Real Roots of the Characteristic Equation. If λ is a root of the characteristic equation 4.7 of multiplicity t then the following are linearly independent solutions of 4.6

$$\{\lambda^n\}_{n=0}^{\infty},\ \{n\,\lambda^n\}_{n=0}^{\infty},\ \{n^2\,\lambda^n\}_{n=0}^{\infty}, \ldots, \{n^{t-1}\,\lambda^n\}_{n=0}^{\infty}.$$

4.2.3 Complex Roots of the Characteristic Equation. Assume that the characteristic equation 4.7 has a complex root $\lambda = a + \imath\, b$ (then it has also a root $a - \imath\, b$). Then

$$\{(a + \imath\, b)^n\}_{n=0}^{\infty} \text{ and } \{(a - \imath\, b)^n\}_{n=0}^{\infty}$$

are **complex** solutions of 4.6.

Let us write the complex numbers above in their geometric form. Then

$$a + \imath\, b = r\,(\cos\varphi + \imath\,\sin\varphi) \text{ and } a - \imath\, b = r\,(\cos\varphi - \imath\,\sin\varphi).$$

Moreover, $(a + \imath b)^n = r^n(\cos n\varphi + \imath\,\sin n\varphi)$ and $(a - \imath b)^n = r^n(\cos n\varphi - \imath\,\sin n\varphi)$. Since any linear combination of solutions of 4.6 is again a solution, we have

$$\{r^n \cos n\varphi\}_{n=0}^{\infty} \text{ and } \{r^n \sin n\varphi\}_{n=0}^{\infty}$$

are lienarly independent solutions of 4.6.

If moreover $a + \imath\, b$ is a root of multiplicity t then

$$\{r^n \cos n\varphi\}_{n=0}^{\infty}, \{n\, r^n \cos n\varphi\}_{n=0}^{\infty}, \ldots, \{n^{t-1}\, r^n \cos n\varphi\}_{n=0}^{\infty}$$

and

$$\{r^n \sin n\varphi\}_{n=0}^{\infty}, \{n\, r^n \sin n\varphi\}_{n=0}^{\infty}, \ldots, \{n^{t-1}\, r^n \sin n\varphi\}_{n=0}^{\infty}$$

are $2t$ linearly independent solutions of 4.6.