4.3 Non-homogeneous Linear Difference Equations with Constant coefficients

4.3.1 A Quasi-polynomial. A function of the form $f(n) = P(n)\,\beta^n$ is called a *quasi-polynomial.*

For example, $f(n) = 3n^2\,(-1)^n$ is a quasi-polynomial, n^2 is a polynomial of degree 2, and $\beta = -1$. At the same time, $f(n) = n + 1$ is a quasi-polynomial; here $f(n) = (n + 1)\,1^n$ and $n + 1$ is a polynomial of degree 1, and $\beta = 1$.

4.3.2 An Estimate of One Solution of a Non-homogeneous Equation. Given a linear equation with constant coefficients

$$a_{n+k} + c_{k-1}\,a_{n+k-1} + \ldots + c_1\,a_{n+1} + c_0\,a_n = b_n. \tag{4.8}$$

Let the right hand side b_n be a quasi-polynomial, $b_n = P(n)\lambda^n$.

Then one of solutions of 4.8 is

$$\hat{a}_n = Q(n)\,n^t\,\beta^n,$$

where $Q(n)$ is a general polynomial of the same degree as $P(n)$ has, and t is the multiplicity of β as a root of the characteristic equation of the associated homogeneous equation.

4.3.3 How to Use the Estimate. Once we have got an estimate $\{\hat{a}_n\}$ of one solution of the non-homogeneous equation 4.8, we substitute it into the equation 4.8 and we get a system of linear equations for unknown coefficients of the polynomial $Q(n)$. If the estimate we made was correct, the system has a unique solution.

4.3.4 Example. Find one solution of the following non-homogeneous equation

$$a_{n+2} + 4\,a_{n+1} - 5\,a_n = 18\,n^2 \tag{4.9}$$

Solution. First of all, we have to find and solve the characteristic equation of the associated homogeneous equation

$$\lambda^2 + 4\,\lambda - 5 = 0.$$

The equation has two solutions: $\lambda = 1$ and $\lambda = -5$.

Further, $b_n = (18\,n^2 + 2)\,1^n$, hence we have the following estimate

$$\hat{a_n} = (An^2 + Bn + C)\,n^1\,1^n;$$

indeed, $An^2 + Bn + C$ is a general polynomial of degree 2, and $\beta = 1$ is a root of the characteristic equation of multiplicity 1. Therefore, we assume

$$\hat{a_n} = An^3 + Bn^2 + Cn.$$

If we substitute the estimate above into 4.9 we obtain

$$A(n+2)^3 + B(n+2)^2 + C(n+2) + 4\,(A(n+1)^3 + B(n+1)^2 + C(n+1)) - -5(An^3 + Bn^2 + Cn) = n^2 + 1.$$

From the above identity of two polynomials we obtain

$$18A\,n^2 + (24A + 12B)\,n + (12A + 8B + 6C) = 18\,n^2 + 2.$$

Hence we get the following system of linear equations for unknown A, B, and C.

$$\begin{array}{ccccccc} 18\,A & & & & & = & 18 \\ 24\,A & + & 12\,B & & & = & 0 \\ 12\,A & + & 8\,B & + & 6\,C & = & 2 \end{array}$$

Thus, $A = 1$, $B = -2$, and $C = 1$.

We have got $\hat{a}_n = n^3 - 2n^2 + n$ is one solution of the non-homogeneous equation 4.8.

4.3.5 A Procedure for Solving Linear Difference Equations with Constant Coefficients. Given a linear difference equation with constant coefficients

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + \ldots + c_1\, a_{n+1} + c_0\, a_n = b_n. \tag{4.10}$$

1) We find a general solution of the associated homogeneous equation

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + \ldots + c_1\, a_{n+1} + c_0\, a_n = 0. \tag{4.11}$$

   It means, we first find roots of the characteristic equation

$$\lambda^k + c_{k-1}\lambda^{k-1} + \ldots + c_1\lambda + c_0 = 0.$$

2) For the quasi-polynomial right hand side, $b_n = P(n)\,\beta^n$ we made an estimate $\hat{a}_n = Q(n)\, n^t\, \beta^n$, where $Q(n)$ is a general polynomial of the same degree as $P(n)$, t is the multiplicity of β as a root of the characteristic equation.

3) We substitute $\hat{a}_n$ into the equation 4.10 and comparing coefficients of the two (equal) polynomials we calculate the unknown coefficients of the polynomial $Q(n)$.

4) General solution of the non-homogeneous equation 4.10 is the sum of a general solution of the associated homogeneous equation and the solution $\hat{a}_n$ from 3).

5) If we are given the initial conditions $a_0,\ a_1,\ \ldots,\ a_{k-1}$, we substitute into the general solution $n = 0,\ n = 1,\ \ldots,\ n = k-1$ and obtain the unknown coefficients from the general solution of the associated homogeneous equation 4.11.