4.4 Asymptotic growth of functions

4.4.1 Symbol $\mathcal{O}$. Given a nonnegative function $g(n)$, we say that a nonnegative function $f(n)$ *is* $\mathcal{O}(g(n))$ if there exist a positive constant c and a natural number n_0 such that

$$f(n) \leq c\, g(n) \quad \text{for all } n \geq n_0.$$

We can consider $\mathcal{O}(g(n))$ to be the class of all nonnegative functions $f(n)$:

$$\mathcal{O}(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \leq c\, g(n) \;\; \forall n \geq n_0\}.$$

4.4.2 Symbol Ω. Given a nonnegative function $g(n)$, we say that a nonnegative function $f(n)$ *is* $\Omega(g(n))$ if there exists a positive constant c and a natural number n_0 such that

$$f(n) \geq c\, g(n) \quad \text{for all } n \geq n_0.$$

We can consider $\Omega(g(n))$ to be the class of all nonnegative functions $f(n)$:

$$\Omega(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \geq c\, g(n) \;\; \forall\, n \geq n_0\}.$$

4.4.3 Remark. It holds that a function $f(n)$ is $\Omega(g(n))$ iff the function $g(n)$ is $\mathcal{O}(f(n))$.

4.4.4 Symbol Θ. Given a nonnegative function $g(n)$, we say that a non negative function $f(n)$ *is* $\Theta(g(n))$ if there exists positive constants c_1, c_2 and a natural number n_0 such that

$$c_1\, g(n) \leq f(n) \leq c_2\, g(n) \quad \text{for all } n \geq n_0.$$

We can consider $\Theta(g(n))$ to be the class of all nonnegative functions $f(n)$:

$$\Theta(g(n)) = \{f(n) \mid \exists c_1, c_2 > 0, n_0 \text{ such that } c_1\, g(n) \leq f(n) \leq c_2\, g(n) \;\; \forall\, n \geq n_0\}.$$

4.4.5 Remark. We have $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is both $\mathcal{O}(g(n))$ and $\Omega(g(n))$.

4.4.6 Symbol small o. Given a nonnegative function $g(n)$. We say that a nonnegative function $f(n)$ *is* $o(g(n))$ if for every positive constant c there exists a natural number n_0 such that

$$0 \leq f(n) < c\, g(n) \quad \text{for all } n \geq n_0.$$

We can consider $o(g(n))$ to be the class of all nonnegative functions $f(n)$:

$$o(g(n)) = \{f(n) \mid \forall\, c > 0\; \exists\, n_0 \text{ such that } 0 \leq f(n) < c\, g(n) \;\; \forall\, n > n_0\}.$$

4.4.7 Remark. A nonnegative function $f(n)$ is $\mathcal{O}(g(n))$ roughly means that the function $f(n)$ does not grow asymptotically more than $g(n)$. On the other hand, to say that a nonnegative function $f(n)$ is $o(g(n))$ roughly means that the function $f(n)$ grows asymptotically less than the function $g(n)$.

4.4.8 Symbol small ω. Given a nonnegative function $g(n)$, we say that a nonnegative function $f(n)$ *is* $\omega(g(n))$ if for every positive constant c there exists a natural number n_0 such that

$$0 \leq c\, g(n) < f(n) \quad \text{for all } n \geq n_0.$$

We can consider $\omega(g(n))$ to be a class of all nonnegative functions $f(n)$:

$$\omega(g(n)) = \{f(n) \mid \forall\, c > 0 \;\text{ there is } n_0 \text{ such that } 0 \leq c\, g(n) < f(n) \;\; \forall\, n > n_0\}.$$

4.4.9 Remark. Roughly speaking, we say that a nonnegative function $f(n)$ is $\Omega(g(n))$ means that the function $f(n)$ grows asymptotically at least as the function $g(n)$. On the other hand, $f(n)$ is $\omega(g(n))$ means roughly that the function $f(n)$ grows asymptotically more than the function $g(n)$.

4.4.10 Notation. Symbols $\mathcal{O}, \Omega, \Theta, o, \omega$ represent classes of functions so we will write $f(n) \in \mathcal{O}(g(n))$; similarly for other symbols $\Omega.\Theta, o, \omega$.

4.4.11 Proposition. Given two nonnegative functions $f(n)$ and $g(n)$, then

- $f(n) \in o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in 0$;
- $f(n) \in \omega(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Justification. Let us write what it meas that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$:

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \text{ it holds } |\frac{f(n)}{g(n)}| < \varepsilon.$$

The fact $|\frac{f(n)}{g(n)}| < \varepsilon$ can be rewritten as $f(n) < \varepsilon\, g(n)$. Denote $c := \varepsilon$, than we get that $f(n)$ is in $o(g(n))$.

3) Analogously, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a$, $a > 0$, means that

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \text{ it holds } |\frac{f(n)}{g(n)} - a| < \varepsilon.$$

Equivalently, $(a-\varepsilon)g(n) < f(n) < (a+\varepsilon)g(n)$. Choose $\varepsilon = \frac{a}{2}$, we get

$$\frac{a}{2}\, g(n) < f(n) < \frac{3a}{2}\, g(n);$$

hence $f(n)$ is $\Theta(g(n))$.

4.4.12 Transitivity. Given three nonnegative functions $f(n)$, $g(n)$ and $h(n)$.

1. If $f(n) \in \mathcal{O}(g(n))$ and $g(n) \in \mathcal{O}(h(n))$, then $f(n) \in \mathcal{O}(h(n))$.
2. If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$, then $f(n) \in \Omega(h(n))$.
3. If $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$, then $f(n) \in \Theta(h(n))$.

4.4.13 Reflexivity. For all nonnegative functions $f(n)$, we have: $f(n) \in \mathcal{O}(f(n))$, $f(n) \in \Omega(f(n))$ and $f(n) \in \Theta(f(n))$.

4.4.14 Proposition. $f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$.

4.4.15 Examples.

1. For every $a > 1$ and $b > 1$, we have

$$\log_a(n) \in \Theta(\log_b(n)).$$

2. It holds that

$$\lg n! \in \Theta(n \lg n).$$

The second part of the above example follows from the following theorem.

4.4.16 Theorem (Gauss). For every $n \geq 1$

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$

Justification. We will use the fact that for every two positive numbers a, b it holds that $\frac{a+b}{2} \geq \sqrt{ab}$.

Let us write $(n!)^2$ as

$$(n!)^2 = n\,(n-1)\,\ldots 2\,1\,1\,2\,\ldots\,(n-1)\,n = \prod_{i=1}^{n}(n-i+1)i.$$

Therefore

$$n! = \prod_{i=1}^{n}\sqrt{(n-i+1)i} \leq \prod_{i=1}^{n}\frac{n+1}{2} = \left(\frac{n+1}{2}\right)^n,$$

since for every i we have $\sqrt{(n-i+1)i} \leq \frac{n-i+1+i}{2}$. We have shown the first estimate.

On the other hand, for every i we have $n \leq (n-i+1)i$, hence $n^n \leq (n!)^2$. Since the both expressions are positive, we can form their square root and get $n^{\frac{n}{2}} \leq n!$.

4.4.17 Theorem. Given a non negative function $f(n)$ which is non decreasing. If $f(\frac{n}{2}) \in \Theta(f(n))$, then

$$\sum_{i=1}^{n} f(i) \in \Theta(n\,f(n)).$$

Justification. The fact that $\sum_{i=1}^{n} f(i) \in \mathcal{O}(n\,f(n))$ is clear: f is non decreasing.

Further, there is a positive constant c such that for sufficiently big n we have $c\,f(n) \leq f(\frac{n}{2})$. Hence

$$\sum_{i=1}^{n} f(i) \geq f(\frac{n}{2}) + \ldots + f(n) \geq \frac{n}{2}\,c\,f(n).$$

This means that $\sum_{i=1}^{n} f(i) \geq \frac{c}{2}\,n\,f(n)$ and therefore $\sum_{i-1}^{n} f(i) \in \Omega(n\,f(n))$.

4.4.18 Remark. The property of the theorem above has for example $f(n) = n^d$ for a natural number $d \geq 1$. On the other hand the function $f(n) = 2^n$ does not fulfill it. To find an asymptotic growth of $\sum_{i=1}^{n} 2^i$ the following method can be used:

Using mathematical induction we show that there is a constant $c > 0$ such that

$$\sum_{i=1}^{n} 2^i \leq c\,2^n.$$

1. Basic step. We know that $\sum_{i=1}^{1} 2^i = 2$ and that $2 \leq c\,2$ for every constant $c \geq 1$.
2. Induction step. Assume that $\sum_{i=1}^{n} 2^i \leq c\,2^n$. Then

$$\sum_{i=1}^{n+1} 2^i = \sum_{i=1}^{n} 2^i + 2^{n+1} \leq c\,2^n + 2^{n+1} = \left(\frac{1}{2} + \frac{1}{c}\right) c\,2^{n+1}.$$

Now, to finish the proof it suffices to find c such that $\frac{1}{2} + \frac{1}{c} \leq 1$. And this is equivalent with the condition that $c \geq 2$.

4.4.19 There is another way how to deal with finding an asymptotic growth of functions of the type

$$\sum_{i=1}^{n} f(i).$$

We show it only for non decreasing functions $f(n)$ (for non increasing you have to change the inequalities):

$$\int_0^n f(x)\,dx \le \sum_{i=1}^{n} f(i) \le \int_1^{n+1} f(x)\,dx.$$

If the integral is improper, it is useful to look for an estimate for $\sum_{i=2}^{n} f(i)$ only.