Week 1
Sets
Discrete Math

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Introduction

Tutorials

- Active participation at tutorials. Homeworks solved in groups — maximal gain 5 points.
- Midterm test during 9th week — maximal gain 15 points.

Exams

- Written test — 80 points.
- Success — at least 40 points from the written test, and the sum of points from homework, from the midterm and the written test must be at least 50 points.

All informations — Moodle

Sets

Examples of sets.

- $S = \{1, 4, 9, 16\}$;
- $T = \{x \mid x = y^2, y \in \mathbb{N}, 0 < y < 5\}$;
- $\mathbb{N}$ – the set of natural numbers;
- $\mathbb{Z}$ – the set of integers;
- $\mathbb{R}$ – the set of real numbers;
- $E = \{m \mid m = 2k, k \in \mathbb{N}\}$ – the set of even natural numbers;
- $O = \{m \mid m = 2k + 1, k \in \mathbb{N}\}$ – the set of odd natural numbers.

Two sets S and T are equal (we write $S = T$) if every element of the set S is an element of the set T and conversely, every element of T is an element of S.

Subsets

Given two sets S and T; S is a subset of T if every element of the set S is also an element of T, we write $S \subseteq T$.

T is not a subset of S if and only if there is an element x for which $x \in T$ and $x \notin S$.

Proposition.
For all sets S, T we have $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

The empty set is a set that contains no element; is denoted $\emptyset$.

Fact.
$\emptyset \subseteq A$ for every set A.

Set Operations

Given two sets A and B

- Union A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$;
- Intersection A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$;
- Difference of two sets A and B (in this order) is the set $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$;
- Difference between an universal set U and A is the complement of A denoted by $\overline{A}$;
- Cartesian product of A and B (we denote it by $A \times B$) is $A \times B = \{(a, b) \mid a \in A, b \in B\}$.
- If $A = B$ then $A \times A$ is an Cartesian power of the set A and we write A^2 instead of $A \times A$, A^3 is the set of all triples of elements of A, etc.

Set Operations

If $A \cap B = \emptyset$, we say that the sets A and B are disjoint sets.

Let A be a set. The power set $P(A)$ of the set A is the set of all subsets of the set A.

Examples.
$P(\emptyset) = \{\emptyset\}$, and $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Fact. The power set $P(A)$ of any set A is nonempty.

Set Operations

Characteristic function of a subset.

A characteristic function χ_A of a subset $A \subseteq U$ is the mapping $\chi_A \colon U \to \{0,1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A; \\ 0 & \text{for } x \in U \setminus A. \end{cases}$$

Fact. There is one-to-one correspondence between subsets of U and characteristic functions, i.e. mappings $U \to \{0,1\}$.

If U has n elements then there are 2^n its subsets.

Cardinality of Sets

A mapping $f\colon A \to B$ is

- injective (or *one-to-one*) if for every two different elements $x, y \in A$ their images $f(x)$, $f(y)$ are also different.
- surjective (or *onto*) if for every element $y \in B$ there exists an element $x \in A$ such that $y = f(x)$.
- a bijection if it is injective and surjective.

Sets A and B have the same cardinality if there is a bijection from A onto B. This fact is denoted by $|A| = |B|$.

Examples

$|\mathbb{N}| = |E|$, $|\mathbb{N}| = |O|$, E is the set of even natural numbers, O is the set of odd natural numbers.

Countable sets

A set A is countable provided it has the same cardinality as the set of all natural numbers $\mathbb{N}$.

If a set A is infinite and not countable then it is *uncountable*.

Fact. A set A is countable if and only if it can be arranged in an injective infinite sequence, (i.e. a sequence which is infinite and where no two elements are equal).

Proposition.

- Any infinite subset of a countable set is again countable.
- If two sets are countable so is their union.
- The Cartesian product of two countable sets is a countable set.
- The union of a countable system of finite disjoint sets is again countable.

Countable sets

Examples.

- The set $\mathbb{Z}$ of all integers is countable.
- The set $\mathbb{Q}$ of all rational numbers is countable.
- The set of all polynomials with integer coefficients is countable.
- The set of all binary words is countable.
- The set of all rational numbers x, $0 < x < 1$ is countable.

Uncountable sets

Cantor Diagonal Method.

Theorem.

The set of all infinite sequences of 0's and 1's is uncountable.

Corollary.

The set of all subsets of natural numbers $\mathbb{N}$, i.e. $P(\mathbb{N})$, is uncountable.