Week 2
Relations
Discrete Math

Marie Demlová
http://math.fel.cvut.cz/en/people/demlova

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Binary Relations

A relation (more precisely a binary relation) from a set A into a set B is any set of ordered pairs $R \subseteq A \times B$.

If $A = B$ we speak about a relation on a set A.

Examples.

- To be a subset. Objects are subsets of a given set U; a subset X is related to a subset Y if X is a subset of Y.
- To be greater or equal. Objects are numbers; a number n is related to a number m if n is greater than or equal to m.
- To be a student of a study group. Objects are first year students and study groups; a student a is related to a study group number K if student a belongs to study group K.
- The sine function. Consider real numbers; a number x is related to a number y if $y = \sin x$.

Operations with Relations

Set operations

Let R and S be two relations from a set A into a set B.

- The intersection of relations R and S is $R \cap S$;
- The union of R and S is $R \cup S$;
- The complement of R is $\overline{R} = (A \times B) \setminus R$.

Inverse Relation.

Given R a relation from A into B. Then the inverse relation of the relation R is R^{-1} from B into A, defined

$$x\, R^{-1} y \quad \text{if and only if} \quad y\, R\, x.$$

Operations with Relations

Composition of Relations.

Given R a relation from A into B and S a relation from B into C. Then the composition $R \circ S$ (sometimes also called the *product*), is the relation from A into C defined by:

$$a\,(R \circ S)\,c \quad \text{iff there is } b \in B \text{ such that } \; a\,R\,b \; \text{ and } \; b\,S\,c.$$

Proposition.

The composition of relations is associative. I.e., if R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D then

$$R \circ (S \circ T) = (R \circ S) \circ T\,.$$

Operations with Relations

Proposition.

The composition of relations is not commutative. It is not the case that $R \circ S = S \circ R$ holds for all relations R and S.

Example. Let A be the set of all people in the Czech Republic. Consider the following two relations R, S defined on A:

$$a\,R\,b \text{ iff } a \text{ is a sibling of } b \text{ and } a \neq b.$$

$$c\,S\,d \text{ iff } c \text{ is a child of } d.$$

Then

$$R \circ S \neq S \circ R.$$

Relations on a Set

Properties of relations on a set.

We say that relation R on A is

- reflexive if for every $a \in A$ it is $a\,R\,a$;
- symmetric if for every $a, b \in A$ it holds that: $a\,R\,b$ implies $b\,R\,a$;
- antisymmetric if for every $a, b \in A$ it holds that: $a\,R\,b$ and $b\,R\,a$ imply $a = b$;
- transitive if for every $a, b, c \in A$ it holds that: if $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$.

Equivalence Relations

A relation R on A is equivalence if it is reflexive, symmetric and transitive.

Given an equivalence relation R on A. An equivalence class of R corresponding to $a \in A$ is the set $R[a] = \{b \in A \mid a\,R\,b\}$.

Example 1.
Then relation R is an equivalence on $\mathbb{Z}$:

$$m\,R\,n \quad \text{if and only if} \quad m - n \text{ is divisible by } 12, \ (m, n \in \mathbb{Z}).$$

For R from Example 1 there are twelve distinct equivalence classes, namely $R[i]$, $i = 0, 1, \ldots, 11$.

Equivalence Relations

Properties of the Set of Equivalence Classes. Let R be an equivalence on A. The set $\{ R[a] \mid a \in A\}$ has the following properties:

- Every $a \in A$ belongs to $R[a]$; so $\bigcup\{ R[a] \mid a \in A\} = A$.
- Equivalence classes $R[a]$ are pairwise disjoint. That is, if $R[a] \cap R[b] \neq \emptyset$, then $R[a] = R[b]$.

Partition. Let A be a non-empty set. A set $\mathcal{S}$ of non-empty subsets of A is a partition of A if the following hold:

1. Every $a \in A$ belongs to some member of $\mathcal{S}$, i.e. $\bigcup \mathcal{S} = A$.
2. The sets in $\mathcal{S}$ are pairwise disjoint. I.e., if $X \cap Y \neq \emptyset$ then $X = Y$ for all $X, Y \in \mathcal{S}$.

Equivalence Relations

Proposition.

Let $\mathcal{S}$ be a partition of A. Then the relation $R_{\mathcal{S}}$ defined by:

$$a\, R_{\mathcal{S}}\, b \quad \text{if and only if} \quad a, b \in X \text{ for some } X \in \mathcal{S}$$

is an equivalence on A.

If we start with an equivalence R, form the corresponding partition into classes of R, and finally we make the equivalence relation corresponding to the partition, we get the equivalence R.

If we start with a partition, then form corresponding equivalence, and finish with the partition into classes of the equivalence, we get the original partition.