Week 3
Relations
Discrete Math

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March 3, 2022

Partial order

A relation R on A is a partial order if it is reflexive, antisymmetric and transitive. A set A together with a partial order is a poset.

Examples.

- $\mathbb{R}$ with $\leq$, an ordinary ordering, is a poset.
- $P(U)$ with $\subseteq$, "to be a subset", is a poset.
- $\mathbb{N}$ with the relation of divisibility defined by $m \mid n$ iff m is a divisor of n (i.e. if $n = k \cdot m$ for some $k \in \mathbb{N}$) is a poset.

Proposition.

If $\sqsubseteq$ is a partial order on a set A, then so is a restriction of $\sqsubseteq$ on any subset $B \subseteq A$.

Partial order

Hasse diagram of a poset.

Let $(A, \sqsubseteq)$ is a poset for a finite set A. The covering relation $\prec$ is a subrelation of $\sqsubseteq$ defined by

$$a \prec b \text{ if and only if } a \neq b \text{ and if } a \sqsubseteq c \sqsubseteq b \text{ then } a = c \text{ or } b = c.$$

The Hasse diagram contains points for all $a \in A$, representing the covering relation where elements smaller are drawn lower than the bigger ones.

Partial order

Smallest element, greatest element.
Given a poset $(A, \sqsubseteq)$.

- $a \in A$ is the smallest element if for every $b \in A$ we have $a \sqsubseteq b$.
- $a \in A$ is the greatest element if for every $b \in A$ we have $b \sqsubseteq a$.

Minimal elements, maximal elements.
Given a poset $(A, \sqsubseteq)$.

- $a \in A$ is a minimal element if for every $b \in A$ we have if $b \sqsubseteq a$ then $b = a$.
- We say that $a \in A$ is a maximal element if for every $b \in A$ we have if $a \sqsubseteq b$ then $b = a$.

Partial order

Facts. Given a poset $(A, \sqsubseteq)$.

- If it has the smallest element, then it is the only minimal element.
- A poset can have more than one minimal but i this case it does not have the smallest element.
- A poset can have no minimal element.
- Any post $(A, \sqsubseteq)$ with a finite set A has at least one minimal and at least one maximal element.

Analogously for the greatest element and maximal elements

Partial order

Linear order, comparable and incomparable elements.

Given a poset $(A, \sqsubseteq)$ and $a, b \in A$. Then a, b are comparable if $a \sqsubseteq b$ or $b \sqsubseteq a$. Otherwise, they are called incomparable.

A partial order on A is a linear order if any two elements of A are comparable.

Partial order

Well-ordering

A partial order $\sqsubseteq$ on A is called well-ordering if any non-empty subset $M \subseteq A$ has the smallest element.

Well-ordering Principle.

Let $\mathbb{N}$ be the set of all natural numbers. Then the ordinary relation $\leq$ "to be smaller or equal to" on $\mathbb{N}$ is a well-ordering.

Well-ordering Principle cannot be either proved or disproved.

Mathematical Induction

Weak form od mathematical induction

Given a property $V(n)$ of natural numbers. Assume that

1. $V(n_0)$ is true;
2. if $V(n)$ holds then $V(n+1)$ holds as well.

Then $V(n)$ is true for any $n \geq n_0$.

Mathematical Induction

Example 1.

Prove using the mathematical induction that for any set U with n elements the set $\mathcal{P}(U)$ has 2^n elements (for any $n \geq 0$).

Example 2.

Derive a formula $\sum_{i=0}^{n} i^2$.

Mathematical Induction

Theorem

The principle of mathematical induction follows from the well-ordering principle.