Week 4
Mathematical Induction
Discrete Math

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Mathematical Induction

Well-ordering.

A partial order $\sqsubseteq$ on A is called well-ordering if any non-empty subset $M \subseteq$ has the smallest element.

Well-ordering Principle.

Let $\mathbb{N}$ be the set of all natural numbers. Then the ordinary relation $\leq$ "to be smaller or equal to" is a well-ordering.

Weak form of mathematical induction.

Given a property $V(n)$ of natural numbers. Assume that

1. $V(n_0)$ is true;
2. if $V(n)$ holds for $n \geq n_0$ then $V(n+1)$ holds as well.

Then $V(n)$ is true for any $n \geq n_0$.

Mathematical Induction

Strong form.

Given a property $V(n)$ of natural numbers. Assume that

1. $V(n_0)$ is true;

2'. if $V(k)$ holds for every $n_0 \leq k < n$ then $V(n)$ holds as well.

Then $V(n)$ is true for any $n \geq n_0$.

Example 2.

Prove by strong mathematical induction the following statement:
Every natural number $n \geq 2$ is a product of one or more primes.

Mathematical Induction

Theorem.

The weak and the strong mathematical induction are equivalent.

Theorem.

The well-ordering principle follows from the strong version of mathematical induction.

Mathematical Induction

Example 3.

Hanoi Towers

Example 4.

Tiling Problem.

A right tromino is a figure consisting in three squares of the same size arranged to a right angle. A deficient board of 2^{2n} squares is a square with one square missing.

Is it possible to cover a deficient boar by trominos for every $n \geq 1$.

Mathematical Induction

Example 5.

Egyptian form of a rational number between 0 and 1

Given a natural number $\frac{p}{q}$, where $0 < \frac{p}{q} < 1$, then there are natural numbers $r_1, r_2, \ldots, r_k$, $r_i \neq r_j$ for $i \neq j$, such that

$$\frac{p}{q} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_k}.$$

Mathematical Induction

Structural induction.

Mathematical induction is used also for constructing sets. Then proving properties of elements of the set us usualy done by mathematical induction which is then called *structural induction*.

Example 6.

Let A be a set of binary words defined inductively by:

- $0 \in A$ and $1 \in A$.
- If $w \in A$ then $0w0 \in A$ and $1w1 \in A$.

Prove that A consists of all binary words of odd length which are palindromes (i.e. words w that are the same as its reverse).