Week 10
Difference Equations
Discrete Math

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Difference Equations, Recursive Equations

Sequences.

A sequence is a mapping from the set of all integers greater or equal to an integer n_0 into the set of all real numbers. Hence

$$\{a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots\} \quad \text{where } a_i \in \mathbb{R}.$$

Linear Difference Equations.

Let $c_i(n)$, $i \in \{0, \ldots, k-1\}$, be functions $\mathbb{Z} \to \mathbb{R}$, $c_0(n)$ not identically zero, and let $\{b_n\}_{n=n_0}^{\infty}$ be a sequence. Then the equation

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = b_n, \;\; n \geq n_0$$

is a linear difference equation of order k (also a linear recursive equation of order k).

Difference Equations, Recursive Equations

Functions $c_i(n)$ are coefficients of the equation, the sequence $\{b_n\}_{n=n_0}^{\infty}$ the right-hand side of the equation.

If $\{b_n\}_{n=n_0}^{\infty}$ is the zero sequence then we speak about homogeneous equation, otherwise the equation is non-homogeneous.

We write a linear difference equation also

$$a_{n+k} + \sum_{i=0}^{k-1} c_i(n)\, a_{n+i} = b_n,\ \ n \geq n_0.$$

Difference Equations, Recursive Equations

Solutions of Linear Difference Equations.

A solution of a linear difference equation is any sequence $\{u_n\}_{n=n_0}^{\infty}$ such that if we substitute u_n for a_n in it we obtain a statement that is valid.

Initial Conditions.

Given a linear difference equation of order k

$$a_{n+k} + c_{k-1}(n)\, a_{n+k-1} + \ldots + c_1(n)\, a_{n+1} + c_0(n)\, a_n = b_n, \;\; n \geq n_0$$

By initial conditions we mean the following system

$$a_{n_0} = A_0,\; a_{n_0+1} = A_1,\; \ldots,\; a_{n_0+k-1} = A_{k-1},$$

where A_i are real numbers.

Difference Equations, Recursive Equations

The Associated Homogeneous Equation.

Given a linear difference equation

$$a_{n+k} + c_{k-1}(n)\,a_{n+k-1} + \ldots + c_1(n)\,a_{n+1} + c_0(n)\,a_n = b_n,\ \ n \geq n_0.$$

Then the equation

$$a_{n+k} + c_{k-1}(n)\,a_{n+k-1} + \ldots + c_1(n)\,a_{n+1} + c_0(n)\,a_n = 0,\ \ n \geq n_0.$$

is the associated homogeneous equation to the equation above.

Difference Equations, Recursive Equations

Proposition.

Given a linear difference equation. Then the following holds:

1. If $\{u_n\}_{n=n_0}^{\infty}$ and $\{v_n\}_{n=n_0}^{\infty}$ are two solutions of non homogeneous equation then $\{u_n\}_{n=n_0}^{\infty} - \{v_n\}_{n=n_0}^{\infty}$ is a solution of the associated homogeneous equation.
2. If $\{u_n\}_{n=n_0}^{\infty}$ is a solution of non homogeneous equation and $\{w_n\}_{n=n_0}^{\infty}$ is a solution of the associated homogeneous equation, then $\{u_n\}_{n=n_0}^{\infty} + \{w_n\}_{n=n_0}^{\infty}$ is a solution of non homogeneous equation.
3. Let $\{\hat{u}_n\}_{n=n_0}^{\infty}$ be a fixed solution of the non homogeneous equation. Then for every solution $\{v_n\}_{n=n_0}^{\infty}$ of it there exists a solution solution $\{w_n\}_{n=n_0}^{\infty}$ of the associated homogeneous equation for which
$$\{v_n\}_{n=n_0}^{\infty} = \{\hat{u}_n\}_{n=n_0}^{\infty} + \{w_n\}_{n=n_0}^{\infty}.$$

Difference Equations, Recursive Equations

Theorem.

Given a homogeneous linear difference equation. Then for the set S of all solutions the following holds:

1. If $\{u_n\}_{n=n_0}^{\infty}$ and $\{v_n\}_{n=n_0}^{\infty}$ belong to S then so does $\{u_n\}_{n=n_0}^{\infty} + \{v_n\}_{n=n_0}^{\infty}$.
2. If $\{u_n\}_{n=n_0}^{\infty}$ belongs to S and α is any real number, then $\{k\ u_n\}_{n=n_0}^{\infty}$ belongs to S as well.

Difference Equations, Recursive Equations

The difference equation

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + \ldots + c_1\, a_{n+1} + c_0\, a_n = b_n,\ n \geq n_0,\ c_i \in \mathbb{R},$$

i.e. coefficients $c_i(n)$ are constant functions, is difference equation with constance coefficients.

Characteristic equation of the equation above is

$$\lambda^k + c_{k-1}\lambda^{k-1} + \ldots + c_1\lambda + c_0 = 0.$$

Any λ satisfying characteristic equation leads to one solution

$$a_n = \{\lambda^n\}_{n=n_0}^{\infty}.$$

Linear Difference Equations with Constant Coefficients

Real roots of characteristic equation.

If λ is a root of the characteristic equation of multiplicity t then the following are linearly independent solutions of its homogeneous equation

$$\{\lambda^n\}_{n=0}^{\infty},\ \{n\,\lambda^n\}_{n=0}^{\infty},\ \{n^2\,\lambda^n\}_{n=0}^{\infty},\ldots,\{n^{t-1}\,\lambda^n\}_{n=0}^{\infty}.$$

Linear Difference Equations with Constant Coefficients

Complex roots of characteristic equation.

If $\lambda = a + \imath\, b$ is a complex root of the characteristic equation of multiplicity t then the following are linearly independent complex solutions of its homogeneous equation

$$\{(a + \imath\, b)^n\}_{n=0}^{\infty} \text{ and } \{(a - \imath\, b)^n\}_{n=0}^{\infty}$$

and the following real solutions

$$\{r^n \cos n\varphi\}_{n=0}^{\infty} \text{ and } \{r^n \sin n\varphi\}_{n=0}^{\infty}$$

Exercises

Exercise 1.
Given a group $(\mathbb{Z}_{17}^{\star}, \cdot, 1)$. Find the order of 2. Is 2 a generating element? Write down $\langle 2 \rangle$ in $\mathbb{Z}_{17}^{\star}$.

Exercise 2.
Given a group $(\mathbb{Z}_{17}^{\star}, \cdot, 1)$. Find all its generating elements.

Exercises

Exercise 3.
Given a group $(\mathbb{Z}_{17}^{\star}, \cdot, 1)$. Find all its subgroups.

Exercise 4.
Given a group $(\mathbb{Z}_{14}^{\star}, \cdot, 1)$.

a) Write down all its elements.
b) Find orders $r(a)$ for all its elements.
c) Is the group a cyclic group?
d) Find all its subgroups.

Exercises

Exercise 5.
Solve the following difference equation

$$5a_n = a_{n+1} + 6a_{n-1},\ n \geq 1,\ a_1 = 9, a_2 = 21. \tag{1}$$

Exercise 6.
Solve the following difference equation

$$a_n = -2n\, a_{n-1} + 3n(n-1)\, a_{n-2},\ a_0 = 1, a_1 = 2 \tag{2}$$

using the substitution $a_n = n!\, b_n$.

Exercises

Exercise 7.
The Fibonacci sequence satisfies

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0, F_1 = 1. \tag{3}$$

Find the formula for F_n.

Exercise 8.
Solve the following difference equation

$$5a_n = a_{n+1} + 6a_{n-1},\ n \geq 1,\ \ a_1 = 9, a_2 = 21.$$