Week 10
Difference Equations

Discrete Math

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Linear Difference Equations with Constant Coefficients

The difference equation

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + \ldots + c_1\, a_{n+1} + c_0\, a_n = b_n,\ \ n \geq n_0,\ \ c_i \in \mathbb{R},$$

i.e. coefficients $c_i(n)$ are constant functions, is difference equation with constant coefficients.

Characteristic equation of the equation above is

$$\lambda^k + c_{k-1}\lambda^{k-1} + \ldots + c_1\lambda + c_0 = 0.$$

Any λ satisfying characteristic equation leads to one solution

$$a_n = \{\lambda^n\}_{n=n_0}^{\infty}.$$

Homogeneous Linear Difference Equations with Constant Coefficients

Real roots of characteristic equation.

If λ is a root of the characteristic equation of multiplicity t then the following are linearly independent solutions of its homogeneous equation

$$\{\lambda^n\}_{n=0}^{\infty},\ \{n\,\lambda^n\}_{n=0}^{\infty},\ \{n^2\,\lambda^n\}_{n=0}^{\infty},\ldots,\{n^{t-1}\,\lambda^n\}_{n=0}^{\infty}.$$

Homogeneous Linear Difference Equations with Constant Coefficients

Complex roots of characteristic equation.

If $\lambda = a + \imath\, b$ is a complex root of the characteristic equation of multiplicity t then the following are linearly independent complex solutions of its homogeneous equation

$$\{(a + \imath\, b)^n\}_{n=0}^{\infty} \text{ and } \{(a - \imath\, b)^n\}_{n=0}^{\infty}$$

and the following real solutions

$$\{r^n \cos n\varphi\}_{n=0}^{\infty} \text{ and } \{r^n \sin n\varphi\}_{n=0}^{\infty}$$

Linear Difference Equations with Constant Coefficients

A Quasi-polynomial.

A function of the form $f(n) = P(n)\,\beta^n$ is a quasi-polynomial.

Non-homogeneous Linear Difference Equations with Constant Coefficients

An Estimate of One Solution of a Non-homogeneous Equation.

Given a linear equation with constant coefficients

$$a_{n+k} + c_{k-1}\, a_{n+k-1} + \ldots + c_1\, a_{n+1} + c_0\, a_n = b_n,$$

where b_n is a quasi-polynomial, $b_n = P(n)\lambda^n$.

We seek one of its solutions of the form

$$\hat{a}_n = Q(n)\, n^t\, \beta^n,$$

where $Q(n)$ is a suitable polynomial of the same degree as $P(n)$, and t is the multiplicity of β as a root of the characteristic equation.

Non-homogeneous Linear Difference Equations with Constant Coefficients

How to Use the Estimate.

Once we have got an estimate $\{\hat{a}_n\}$ of one solution of the non-homogeneous equation, then

- we substitute it into the non-homogeneous equation,
- we get a system of linear equations for unknown coefficients of the polynomial $Q(n)$,
- if the estimate is correct, the system has a unique solution.

A Procedure for Solving Linear Difference Equations with Constant Coefficients.

The procedure

1) We calculate the general solution of the associated homogeneous equation.

2) For $b_n = P(n)\,\beta^n$ we made an estimate $\hat{a}_n = Q(n)\,n^t\,\beta^n$, where $Q(n)$ is a general polynomial of the same degree as $P(n)$, t is the multiplicity of β as a root of the characteristic equation.

3) $\hat{a}_n$ is substituted into the non-homogeneous equation; comparing coefficients of the two (equal) polynomials we get the coefficients of $Q(n)$.

A Procedure for Solving Linear Difference Equations with Constant Coefficients.

4) General solution of the non-homogeneous equation is the sum of a general solution of the associated homogeneous equation and the solution $\hat{a}_n$ from 3).

5) If initial conditions $a_0, a_1, \ldots, a_{k-1}$ are given, we substitute into the general solution $n = 0$, $n = 1$, ..., $n = k - 1$ and obtain the unknown coefficients from the general solution of the associated homogeneous equation.

Examples

Example 1.

Solve the following difference equation

$$a_n = -2n\, a_{n-1} + 3n(n-1)\, a_{n-2},\ \ a_0 = 1, a_1 = 5 \qquad (1)$$

using the substitution $a_n = n!\, b_n$.

Examples

Example 2.

Form a linear differnce equation which has the following general solution

$$\{a_n\}_{n=0}^{\infty} = \{\alpha + \beta\,(-1)^n + \gamma n(-1)^n\}_{n=0}^{\infty};\ \alpha, \beta, \gamma \in \mathbb{R}.$$

A Procedure for Solving Linear Difference Equations with Constant Coefficients.

Example 3.

Solve the following difference equation

$$a_n - 5\,a_{n-1} + 6\,a_{n-2} = 12n, \quad a_0 = 25, a_1 = 36 \tag{2}$$

Examples

Example 4.

Solve the following difference equation

$$a_{n+2} - 3a_{n+1} + 2a_n = 6 \cdot 2^n,\ a_0 = 2, a_1 = 6.$$