Week 12
Asymptotic Growth of Functions

Discrete Math

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Asymptotic Growth of Functions

Symbol $\mathcal{O}$.

Let $g(n)$ be a non-negative function. A non-negative function $f(n)$ is $\mathcal{O}(g(n))$ if there is a positive constant c and a natural number n_0 such that

$$f(n) \leq c\, g(n) \quad \text{for every } n \geq n_0.$$

$\mathcal{O}(g(n))$ is considered as a class of non-negative function $f(n)$:

$$\mathcal{O}(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \leq c\, g(n) \;\; \forall n \geq n_0\}.$$

Example. $3n \in \mathcal{O}(n)$, $3n \in \mathcal{O}(n^2)$.

Asymptotic Growth of Functions

Symbol Ω.

Let $g(n)$ be a non-negative function. A non-negative function $f(n)$ is $\Omega(g(n))$ if there is a positive constant c and a natural number n_0 such that

$$f(n) \geq c\,g(n) \quad \text{for every } n \geq n_0.$$

$\Omega(g(n))$ is considered as a class of non-negative function $f(n)$:

$$\Omega(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \geq c\,g(n) \;\; \forall\, n \geq n_0\}.$$

Example. $3n \in \Omega(n)$, $3n \in \Omega(\sqrt{n})$.

Fact. $f(n)$ is $\Omega(g(n))$ iff $g(n)$ is $\mathcal{O}(f(n))$.

Asymptotic Growth of Functions

Symbol Θ.

Let $g(n)$ be a non-negative function. A non-negative function $f(n)$ is $\Theta(g(n))$ if there are constants c_1, c_2 and a natural number n_0 such that

$$c_1\, g(n) \leq f(n) \leq c_2\, g(n), \quad \forall\, n \geq n_0.$$

$\Theta(g(n))$ is considered as a class of non-negative function $f(n)$:

$$\{f(n) \mid \exists c_1, c_2 > 0, n_0; \quad c_1\, g(n) \leq f(n) \leq c_2\, g(n), \quad \forall\, n \geq n_0\}.$$

Fact. $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is $\mathcal{O}(g(n))$ and $\Omega(g(n))$.

Asymptotic Growth of Functions

Proposition. $f(n) \in \Theta(g(n))$ iff $g(n) \in \Theta(f(n))$.

Examples.

- For every $a > 1$ and $b > 1$ we have

$$\log_a(n) \in \Theta(\log_b(n)).$$

- The logarithm with base 2 is usually denoted by lg, i.e. $\lg(n) = \log_2(n)$. It holds that

$$\lg n! \in \Theta(n \lg n).$$

Asymptotic Growth of Functions

Theorem (Gauss).

For every $n \geq 1$ it holds that

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$

Asymptotic Growth of Functions

Symbol small o. Given a non-negative function $g(n)$, we say that a non-negative function $f(n)$ *is* $o(g(n))$ if for every $c > 0$ there is $n_0 \in \mathbb{N}$ such that

$$0 \leq f(n) < c\, g(n) \quad \text{for all } n \geq n_0.$$

$$o(g(n)) = \{f(n) \mid \forall\, c > 0\, \exists\, n_0 \text{ such that } 0 \leq f(n) < c\, g(n) \;\; \forall\, n > n_0\}.$$

Asymptotic Growth of Functions

Symbol small ω. Given a non-negative function $g(n)$, we say that a non-negative function $f(n)$ *is* $\omega(g(n))$ if for every $c > 0$ there is $n_0 \in \mathbb{N}$ such that

$$f(n) > c\, g(n) \quad \text{for all } n \geq n_0.$$

$$\omega(g(n)) = \{f(n) \mid \forall\, c > 0\ \exists\, n_0 \text{ such that } f(n) > c\, g(n) \leq 0\ \ \forall\, n > n_0\}.$$

Asymptotic Growth of Functions

Proposition. Given two non-negative functions $f(n)$ and $g(n)$, then

- $f(n) \in o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$;
- $f(n) \in \omega(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a$, $a \in \mathbb{R}$, $a \neq 0$, then $f(n) \in \Theta(g(n))$.

Asymptotic Growth of Functions

Transitivity. Given three non-negative functions $f(n)$, $g(n)$ and $h(n)$.

- If $f(n) \in \mathcal{O}(g(n))$ and $g(n) \in \mathcal{O}(h(n))$, then $f(n) \in \mathcal{O}(h(n))$.
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$, then $f(n) \in \Omega(h(n))$.
- If $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$, then $f(n) \in \Theta(h(n))$.

Reflexivity.
For all non-negative functions $f(n)$, we have: $f(n) \in \mathcal{O}(f(n))$, $f(n) \in \Omega(f(n))$ and $f(n) \in \Theta(f(n))$.

Estimations of functions

Proposition.

Given non negative function $f(n)$ which is non decreasing. If $f(\frac{n}{2}) \in \Theta(f(n))$ then

$$\sum_{i=1}^{n} f(i) \in \Theta(n\,f(n)).$$

Remark. The above property has e.g. $f(n) = n^d$ for natural number $d \geq 1$, but the function $f(n) = 2^n$ not.

For $\sum_{i=1}^{n} 2^i$ we can use mathematical induction and show

$$\sum_{i=1}^{n} 2^i \leq c\,2^n.$$

Estimations of functions

Another possibility

Let $f(n)$ be a positive increasing function. Then

$$\int_0^n f(x)\,dx \leq \sum_{i=1}^{n} f(i) \leq \int_1^{n+1} f(x)\,dx.$$

Let $f(n)$ be a positive decreasing function. Then

$$\int_1^{n+1} f(x)\,dx \leq \sum_{i=1}^{n} f(i) \leq \int_0^{n} f(x)\,dx.$$