1 Asymptotic growth of functions

1.1 Decide whether the following holds:

(a) $2^{n+1} \in \Theta(2^n)$,

(b) $2^{\frac{n}{2}} \in \Theta(2^n)$.

1.2 Given functions

$$5^n,\ n!,\ n^{\frac{1}{\ln\ n}},\ e^n,\ \ \ln\ (n!),\ \ \ln\ \ \ln n,\ n\log_2 n,\ \ \ln\ (n^{\ \ln\ n}),\ \sqrt{n},\ \frac{1}{n}.$$

Sort these functions in $f_1, f_2, \dots$, such that $f_i \in \mathcal{O}(f_{i+1})$ where $i = 1, 2, \dots$ State all the pair of functions f, g for which $f \in \Theta(g)$.

1.3 Prove or disprove:

(a) If for two non negative functions we have $f(n) \in \mathcal{O}(g(n))$ then $2^{f(n)} \in \mathcal{O}(2^{g(n)})$.

(b) For every non negative function $f(n)$ we have $f(n) \in \mathcal{O}(f(n)^2)$.

1.4 Given the function $f(n) = \sum_{i=1}^{n} \frac{1}{i}$. Find a simplest function $g(n)$ for which $f(n) \in \Theta(g(n))$.

1.5 Given two functions $f(n) = n^{\ \ln\ n}$ and $g(n) = (\log_2 n)^n$. Decide whether $f(n) \in \mathcal{O}(g(n))$, or $g(n) \in \mathcal{O}(f(n))$ or neither of them.

1.6 Find an upper bound for

$$\sum_{k=1}^{n} \frac{k}{3^k}.$$

Hint: Use a suitable geometric series.

1.7 Prove the following assertion: *Given three non negative functions $f(n)$, $g(n)$, and $h(n)$ for which there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$*

$$g(n) \leq f(n) \leq h(n).$$

Assume that $g(n) \in \Omega(k(n))$ and $h(n) \in \mathcal{O}(k(n))$. Then $f(n) \in \Theta(k(n))$.

Problems to be solved before next tutorial

1.8 Prove (using the definition) that the function $f(n) = 2 \cdot 5^n + 1000\, n^4$ satisfies $f(n) \in \Theta(5^n)$.

1.9 Given three non negative functions $f(n)$, $g(n)$ and $h(n)$. Prove:

If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.

1.10 Prove the following assertion: *Given non negative functions $f_1(n)$, $f_2(n)$ and $h(n)$ such that*

$$f_1(n) \in \Theta(h(n)),\ f_2(n) \in \Theta(h(n)\ \log_2 n),$$

then $(f_1 + f_2)(n) \in \Theta(h(n)\ \log_2 n)$.