2 Asymptotic growth of functions

2.1 Solving homework from the last tutorial

(a) Prove (using the definition) that the function $f(n) = 2 \cdot 5^n + 1000n^4$ satisfies $f(n) \in \Theta(5^n)$.

(b) Given three non negative functions $f(n)$, $g(n)$ and $h(n)$. Prove:

If $f(n) \in \mathcal{O}(g(n))$ and $g(n) \in \mathcal{O}(h(n))$ then $f(n) \in \mathcal{O}(h(n))$.

(c) Prove the following assertion: *Given non negative functions $f_1(n)$, $f_2(n)$ and $h(n)$ such that*

$$f_1(n) \in \Theta(h(n)),\ f_2(n) \in \Theta(h(n)\ \log_2 n),$$

then $(f_1 + f_2)(n) \in \Theta(h(n)\ \log_2 n)$.

2.2 An example from the last tutorial

Given the function $f(n) = \sum_{i=1}^{n} \frac{1}{i}$. Find a simplest function $g(n)$ for which $f(n) \in \Theta(g(n))$.

2.3 Prove or disprove: *If for two non-negative functions $f(n)$ and $g(n)$ it holds $f(n) \in \Theta(g(n))$ a $g(n) \in \Omega(1)$, then for every constant $k > 0$ we have $f(n) + k \in \Theta(g(n))$.*

2.4 Prove or disprove: *Assume that for non negative functions $f(n)$ and $g(n)$ we have $f(n) \in \Theta(g(n))$ and $g(n) \in \Omega(1)$. Then*

$$\sum_{i=1}^{n} f(i) \in \Theta(\sum_{i=1}^{n} g(i)).$$

2.5 State the asymptotic growth:

(b) $\sum_{i=1}^{n} (8\frac{i}{2^i})$.

(c) $\sum_{i=1}^{n} (i^4\ \log_2^3 i + i^3\ \log_2^9 i)$.

Problems to be solved before next tutorial

2.6 Prove or disprove: *Assume that for non negative functions $f(n)$ and $g(n)$ we have $f(n) \in \Theta(h(n))$ and $g(n) \in \omega(h(n))$. Then*

$$f(n) + g(n) \in \Theta(g(n)).$$

2.7 State the asymptotic growth:

(a) $\sum_{i=0}^{n} (7i^3 + 3i^2 + 16)$.

(b) $\sum_{i=1}^{n} (5^i + i^{100})$.

Additional example

2.8 Prove or disprove: Assume that non negative function $f(n)$ satisfies $f(n) \geq n$ for $k \geq 2$. Then

$$f(n) \cdot k^{f(n)} \in 2^{\mathcal{O}(f(n))}.$$

The class $2^{\mathcal{O}(f(n))}$ of all non negative functions $g(n)$ for which there is $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $g(n) \leq 2^{c\,f(n)}$. (Another formulation: $g(n) = 2^{h(n)}$ where $h(n) \in \mathcal{O}(f(n))$.)