

## 12<sup>th</sup> SEFI Maths Working Group Seminar

*Proceedings*

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- to contribute to the development and the improvement of engineering education and of the position of the engineering professionals
- to provide appropriate services and to promote information about engineering education
- to improve communication and exchanges between teachers, researches and students in the various European countries
- to develop co-operation between educational engineering institutions and establishment of higher technical education
- to promote co-operation between industry and those engaged in engineering education
- to act as an interlocutor between its members and other societies or organizations
- to promote the European dimension in higher engineering education

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## Preface

The Mathematics Working Group (*SEFI MWG*) was founded in 1982. Its main purpose is to establish, coordinate and support the mathematical needs of future engineers on an international basis. From the outset, one of the most important aspects has been the exchange of ideas and experiences in teaching mathematics to future engineers. Regular seminars have been organised to enable this to take place.

The Twelfth European Seminar of the *SEFI MWG* was held on June 14–16, 2004, at the Vienna University of Technology. On Sunday evening, June 13, an informal get-together party (including a buffet dinner) was held in the historical Festival Hall of the university. Here the participants had the opportunity to rekindle their friendships and meet new people.

Round table discussions were a key component of this year's seminar. Discussions were organised on each of the three days of the conference. There were two parallel groups before lunch and an afternoon plenary to summarize the discussion.

Shirley Booth started the first day with a stimulating plenary lecture *Learning and Teaching for Understanding Mathematics*. After this talk Leslie Mustoe and Carl-Henrik Fant organised the round table discussion on *What Are the Key Issues in Teaching Mathematics for Understanding?* In the afternoon Leslie summarized these discussions.

Milton Fuller opened the second day with a plenary talk *Mathematics in Engineering Education in Australia*. Two round table groups discussed *Innovative Ways in Teaching Engineering Mathematics and the Impact of the Bologna Declaration*. These discussions were lead by Marie Demlová and Daniela Velichová. Following the afternoon summary, our host, Hans Kaiser, organised a city tour of Vienna in the afternoon for which he acted as guide. He was an enthusiastic and informative guide and we learned a great deal about the history, architecture and music of Vienna. The excursion ended with an excellent dinner in *Heuriger Mayer Am Pfarrplatz*.

The last day of the seminar was devoted to the *How Should We Assess Engineering Mathematics?* The plenary talk was delivered by Duncan Lawson who reported the results of a questionnaire prepared by the *SEFI MWG* Steering Committee and raised some emerging issues related to assessment. The questionnaire was aimed at investigating the various methods of assessment of engineering mathematics used throughout Europe. Duncan Lawson and Carol Robinson organised the two parallel round table discussions.

In addition to the plenary talks and round table discussions, throughout each of the three days there were a number of contributed talks relevant to the theme of the day. The Festival Hall was also used for the display, throughout the duration of the conference, of a range of posters.

Thanks are due to all who contributed to a highly successful seminar, especially to Hans Kaiser, our host, and Martina Lederhilger-Widl who was a highly efficient and very friendly local organiser. They made the seminar a very pleasant one to attend. I would like to thank the speakers, delegates and all who participated in our discussions.

*Marie Demlová*

Future seminars are planned at Kongsberg, Norway, 2006, and Loughborough, England, 2008. We wish the future organisers success in continuing our tradition of friendly and stimulating seminars.

Editors: Marie Demlová  
Duncan Lawson

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Published by Vydavatelství ČVUT, Prague, 2004

ISBN 2-87352-048-5

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# Notes on Round Table Discussions

**Duncan Lawson**

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## Introduction

An important aspect of the 12<sup>th</sup> SEFI Mathematics Working Group in June 2004 was the three round table discussions. The delegates were divided into two groups (different groups each time) to address a range of key topics related to the main themes of the conference. What follows is a summary of the main ideas that delegates put forward during these discussions.

## Round Table One

### **Theme—What are the Key Issues in Teaching Engineering Mathematics for Understanding?**

The discussion was stimulated by posing the following questions:

- What do we want students to understand?
- What do we want students to do with their understanding?
- What is the purpose of teaching?
- What are the goals for the lecturers and the students?

It was agreed that there are a range of different things we want students to understand including concepts, techniques, strategies and processes. A key element is that students must be able to understand the language of mathematics and how to use it as a precise means of communication. We want them to be able to use their understanding by applying it in appropriate situations, using it to develop material (primarily in other aspects of engineering) and, in an ideal world, to enable them to pursue an interest in a mathematical approach to engineering. In one sense, the main way we want students to be able to utilise their understanding is to be able, in the future as practising engineers, to cope with unpredictable situations.

We must recognise that there are different levels of understanding and acknowledge that understanding is something that matures over time. A student might have a satisfactory understanding of a concept now, enabling him/her to apply it in appropriate situations. But in a few years time this understanding might have matured to a deeper level, enabling the student to use that concept in a new situation.

The primary purpose of teaching is to facilitate learning. In order to do this we need to know what students do and do not already understand. The vast majority

of undergraduate groups these days have a diverse prior experience of mathematics. This includes a range of misconceptions which must be addressed or they will prevent future understanding. Good teaching identifies the misconceptions and aims to replace them with sound understanding.

We must acknowledge that students' goals will often differ from those of the lecturer. The lecturer may take a higher, longer-term view with the goal being to develop a level of understanding which will serve the student well in the future when they are practising engineers. Often the students' goals are much more short-term and related to passing the current unit of study. Assessment is frequently the main driver of student activity. There was widespread agreement that there is a growing lack of motivation amongst students, particularly of learning subjects which do not appear directly relevant (such as mathematics). This lack of motivation almost guarantees a surface approach to learning. This may be countered to some extent by integration of mathematics with other engineering topics thereby showing its relevance and the use of project work allowing the students to wrestle with realistic problems in which they need to call upon mathematical skills in order to achieve the project aims.

## **Round Table Two**

### **Theme—Innovative Ways of Teaching Engineering Mathematics and the Impact of the Bologna Declaration**

The discussion on was stimulated by posing the following questions:

- Is the approach needed in large institutions different from that in small ones?
- What are the best (or just good) ways of using computers in teaching?
- Should we continue to use traditional ways of teaching?

It was agreed that the first semester is a crucial time for securing student engagement. However, it was common, even in smaller institutions, for students to be taught in large lecture groups. Typically group sizes reduce as the student progresses through their course and there is greater specialisation. However, the use of large groups in the early stages of the course can be counter-productive. Large groups create anonymity and give students a feeling that they will not be missed if they do not attend (and in truth often they are not). Missing lectures is the start of a vicious circle. Students are then reluctant to attend tutorials because they have fallen behind and are embarrassed to show this. Smaller groups throughout can address this but these are not economic. One engineering department at Loughborough University is reported to be trialling the use of swipe cards to record attendance. However, attendance monitoring on its own is of little use—there has to be a structure in place to follow up and counsel those students who are attending infrequently.



The delegates agreed that advances in IT have created many new possibilities in teaching. However there was also agreement that using IT was often seen as a way of reducing costs but that this is not the case. The costs of developing good quality materials are high and whilst there may be fewer traditional staff activities, new ones are required to support the process of learning using computers. There is a need for more extensive research into the effect of IT and e-learning in Higher Education. It is essential that we avoid the mistakes that were made in schools with the introduction of calculators.

Two final points in this discussion were that it is good practice to use a wide range of methods when teaching for understanding and sharing of experience, particularly in the use of computers and e-learning, is very important.

The discussion on the Bologna Declaration simply asked delegates to relate either their institution's or their country's approach to its implementation. A wide range of different positions emerged. It seems as though the countries which have only recently joined the EU have adopted the Bologna principles much more quickly than others. Although in some of the older EU countries individual institutions (such as Chalmers in Sweden) have taken the decision to implement Bologna unilaterally. Where there has been significant change a common effect has been a reduction in the amount of mathematics in engineering courses.

## Round Table Three

### Theme—How should we Assess Engineering Mathematics?

The discussion was stimulated by posing the following questions:

- Since written examinations bear no resemblance to practising as an engineer, are they an appropriate form of assessment?
- Are oral examinations worth the high staff resource they require?
- Why are take away assignments not used more often?
- How appropriate is the use of learning outcomes and assessment criteria in engineering mathematics?

In both groups there was strong support for written examinations as an important component of the assessment regime. It was observed that written examinations have a number of advantages such as security and having a smaller staff time requirement. Written examinations are particularly useful for assessment at the first two levels of learning described in Booth's plenary lecture (i.e. knowledge and carrying out routine calculations). It was also felt that it was possible to assess understanding using appropriate questions, although this was thought to become harder as students progress through a course. Decreasing the number of examinations in the later years of a course was thought to be a good idea—open-ended tasks and project work are better forms of assessment later in the course. The fact that an examination bears

no resemblance to anything in the working life of an engineer was not seen to be a problem as engineering studies is not like working life.

It was recognised that written examinations can have disadvantages. For example, if examinations are at the end of a semester with just a short revision prior to the examination, then this is a recipe for promoting surface learning. Regular formative examinations throughout the semester with an end of semester summative examination was thought to be a better approach. It was agreed that time constrained written examinations were not good for exploring the students' ability to 'have new thoughts'.

There was a difference of opinion about the value of oral examinations; this correlated to the home country of the speaker. Some felt that oral examinations are the best form of assessment, giving the examiner the opportunity to fully explore the student's understanding. However others pointed out that oral examinations take a great deal of staff time, there can be problems of consistency between examiners and there is no opportunity for post-verification. These factors militate against the use of oral examinations with large first year groups. However, it was agreed that there could be a place, later in the course when there are smaller groups because of specialising, for oral examinations to attempt to assess higher level cognitive outcomes.

The value of take away assignments was recognised as giving the opportunity to present students with larger, more open-ended problems and also the opportunity for them to work in groups. However, reservations were expressed about the lack of security and not knowing whether students had copied from each other. A number of ways to address this were suggested. These included giving take away assignments a low weighting in the overall assessment package, making take away assignments a qualifier for the final examination but not a contributor to the final mark and setting different projects to different student groups. It was noted that the last option increased the amount of staff effort required.

There was a feeling that Learning Outcomes are often an administrative device that does not reflect actual practice. In the overwhelming majority of assessment regimes, we do not actually require students to 'pass' every outcome but give a module/unit pass on a mark of 40% overall (where there is a choice of questions students may avoid certain topics and even where there is no choice the aggregating of marks across the paper means we cannot guarantee a certain standard in each area). It was noted that this does mean that students can progress with large gaps in their knowledge. It was agreed that it was important that students were informed of 'the rules' (i.e. what is needed in order to pass) at the start, but it was felt that meaningful generic assessment criteria were very difficult to produce because of the marks aggregation discussed earlier. It was suggested that showing students marked sample solutions gave them an indication of how the marks would be allocated and therefore showed them what was needed to pass.

# Plenary Talks

# Learning and Teaching for Understanding Mathematics

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This paper is an attempt to analyse and cast light on learning and teaching mathematics for understanding. It draws on empirical research and theoretical developments from the field of pedagogy, based on studies of learning and teaching mathematics in an engineering context (Booth, 1993; Booth, 1994; Marton & Booth, 1997; Bowden & Marton, 1998).

The empirical research focuses on learning considered through the students' experience of learning, teaching as supporting learning, and understanding as laying the grounds for an unknown future. There are two major theoretical developments concerned. On the one hand, there is learning and the pedagogy of awareness (Marton & Booth, 1997; Booth, in press), which relates the student's comprehension of features of problems and situations they are faced with to the experience of the pedagogical situation in which they meet them. On the other hand, there is curriculum development for capabilities, a term coined for the sorts of understanding which lay the grounds for coping with future unknown problems and situations.

## Learning and knowledge

First, it is important to consider what we understand by knowledge in order to understand what we mean by learning and teaching for understanding. There are a number of ways of characterising knowledge, mainly thanks to two thousand years of philosophical debate on the theme. The two dominant schools of thought can be coarsely described as rationalist and empiricist, the former locating knowledge primarily in the brain or head, with rational thought processes as the means of producing knowledge, and the latter seeing objects in the world as the prime source of knowledge, which humans can never fully comprehend but can come to terms with through experience of the world.

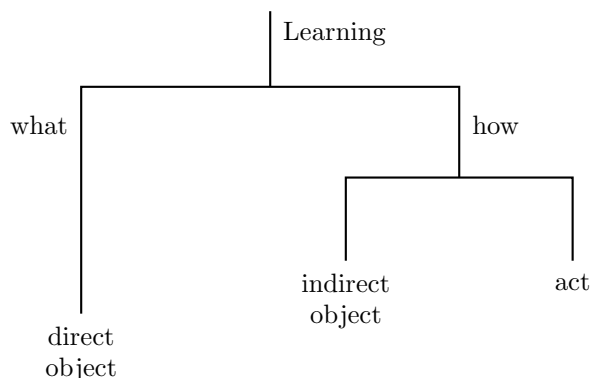
The rationalist school of thought can be seen in today's cognitivist programme of psychology research where the computer is a metaphor for human cognition, and

algorithms and data structures are used to describe learning and memory (e.g. Norman, 1982). The context for learning, in the purest form, is ignored as irrelevant, at least to the models produced. The empiricist school is recognised in the behaviourist movement which was dominant in education in the middle of the 20th century and which led to the many attempts to make teaching more effective through “programmed learning”. That was based on the notion that knowledge could be broken down to a logical sequence of component parts and students who demonstrated that they had mastered one part could go on to the next, mastery consisting of being able to respond correctly to relevant questions (e.g. Skinner, 1968). The mind is ignored as irrelevant, or at best unknowable to the teacher, and only the correct behavioural response to a given stimulus is of interest.

This paper is based in neither of those traditions, but one which can be called phenomenographic, where knowledge is characterised as being a relation between the person and object or phenomenon (Marton & Booth, 1997). Focus shifts from either mind or object to the relation between mind and object. We no longer focus on what happens in the brain but what the individual is doing, with respect to the object of knowledge, and even with respect to the pedagogical context in which she is situated and the socio-cognitive and collaborative context of fellow learners and teachers. Learning is seen as the individual coming to new ways of conceptualising, comprehending, seeing or understanding the phenomenon under study; coming to see new features and relate them to one another and to the whole, as well as to the wider world. This is essentially experiential in as much as coming to experience the phenomenon in a qualitatively new way is seen as the ultimate form of learning, and achieving the ways of understanding that are intended in a programme of education is seen as the goal of teaching.

The empirical educational research tradition of phenomenography, which informs this paper, is grounded in this view of knowledge. One of its most important precepts is that there is variation in the ways people experience phenomena they meet in their worlds, and that this variation can be analysed and described in terms of a small number of qualitatively different categories. Among these categories, teachers can identify the features that are important for current understanding possibly not as comprehensive as the teacher’s own understanding but adequately powerful for current concerns. And by being able to locate this form of understanding in the context of the categories the teacher is also able to identify ways of going further to more sophisticated understanding and to ensure that current understanding also contains the germs of even more powerful ways of understanding for future needs.

In summary, what I mean by learning for the rest of this paper can be summarised as coming to experience something in a qualitatively new and more powerful way, so that it can be accomplished in different circumstances, in different ways, and facilitate doing altogether new things. My choice of words here will become apparent later in the paper. But it cannot be denied that this is a good goal for higher education, irrespective of the field of knowledge we are considering.



**Figure 1.** A model of the experience of learning analysed as content (what or direct object), approach (how), in its turn analysed into act and driving force (indirect object).

Two models will be used in this paper to analyse and describe the experience of learning and teaching. The first relates the “what” the “how” and the “why” of individual learning (Figure 1). “What” is learned is what the teacher is generally most interested in, and it is here that the mathematics returns to our picture. The mathematics teacher has goals for the students, generally expressed in terms of principles, concepts, constructs and skills that together go to make up the content of a course or module of study. This is the domain of the mathematics teacher rather than the educational researcher, though there is much to inform the teacher when it comes to the ways in which their students might be comprehending it all, in particular when it comes to “threshold concepts and troublesome knowledge” (Mayer & Land, 2003).

The “How” of learning relates to the ways in which students go about their tasks of learning, as they are set by teachers: problems, exercises, computer modelling or whatever, whether individual tasks or group tasks. And the “Why” of learning here means the driving force behind their ways of tackling these tasks, derived from their history of study, their understanding of the current situation, and its perceived structure of relevance.

Two studies of how students experience their own approaches to learning mathematics and one important factor in what drives them to do what they do will now be described.

## Variation in approaches to learning mathematics

A study was carried out some years ago at Chalmers University of Technology (Booth, 1994; Booth, 1993), in part to see if first-year students who had gone through a particular form of teaching (explorative mathematics) with the aim of achieving “mathematical power” differed in their views of mathematics and learning it from those who had gone through a more traditional form of education. Representatives of the two groups were interviewed at length, asked to tackle problems and explain

them, and given the opportunity to describe how they went about their studies and why. Their responses to such openings as “Tell me, how do you go about revising for examinations?”, and “What do you do when you get stuck on something you need to understand?”, as well as more straightforward questions such as “What do you mean by mathematics, and learning mathematics?”, were analysed, resulting in two phenomenographic sets of descriptive categories.

The first (Table 1) tells of qualitatively different ways of approaching learning and learning tasks the “how” of our model. The results have been categorised in terms of an overriding strategy for learning, each with an intention and a goal for learning. The first, “Just learning” carries with it the intention of learning the content, as intended by the teacher, so that it is known for future use as needed. It is unquestioned, unrefined and unrelated to anything other than the current course of study with its demands and tasks—largely aimed at the looming examination. A slightly more refined form of this is the second category, “Doing examples”, where the intention is to become proficient at doing examples so that examples of a similar type can be done when needed. Quotes that partially illustrate these two categories follow.

**Table 1.** The variation of ways of approaching learning

Strategy	Intention	Goal
<i>Just learning</i>	To learn the content	To know the content for use when needed
<i>Doing examples</i>	(a) To be able to concretise, generalise and understand the theory (b) To become proficient at doing the examples	To understand the content and be able to make use of it To know the content and be able to use it
<i>Unstudied reflection</i>	Let areas of difficulty rest and mature, then go back to them Discuss difficulties with friends	To understand the content and be able to take it for granted in later use
<i>Studied reflection</i>	To be able to take different perspectives on problems To relate content to the world outside mathematics	To be able to use mathematics to solve problems To understand how mathematics applies to other situations

I Learning mathematics, what is it actually, to learn mathematics?

X1 For me, well, it’s drumming it in, the methods. I really have to sit down and do examples. I have to do masses of examples of the same type, . . . so that it really sticks. I might sit with the book, maybe sit and look back for an example where it did the same thing. . . then I can do it as well. . . If you’re going to

remember it, you need... yes, a lot of examples of the same type. That's true all the way. Then it's a question of learning by heart, parrot-fashion... proofs and things [...] things like that... they usually go in and stick but then they run away when they're not needed any more... they don't stick

and

I To learn mathematics, what does that mean actually?

V1 By trying to understand, first understand the theory, first and foremost, and then going on to work out examples on the theory.

...

I If you meet something new, how do you usually try to get it clear?

V1 Well, I try to look at it all, go to the lectures and then look at the notes on the lecture. And later I look in the book it's usually a bit harder there. And then I do some examples.

Both of these contain expressions of both categories (I want to emphasise that this research does not categorise individual participants but only the ways in which phenomena are voiced, which might well be in more than one way by one individual participant).

X1 wants it to "really stick", to avoid it "running away" by employing "parrot-fashion" learning techniques. This is an important feature of what I mean by "just learning". X1 also "drums in the methods" by doing examples, and works to remember by "doing a lot of examples", which also typifies what I mean by "Doing examples (for proficiency)". V1, similarly, "works out examples on the theory" and "does some examples" to consolidate what has been heard and read, again referring to examples as a means of becoming proficient in doing examples. But V1's statement also bears the germ another intention and goal with "Doing examples", namely, in order to understand content and be able to use it. Maybe, although he does not say so, he is doing examples in order to understand the theory, and in order to cast light on what he has heard and read, and that is qualitatively different use of examples for learning. I will return to this category shortly. The remaining two categories both have an element of reflection: the first unstudied or spontaneous, and the second studied, or strategic. When V8 tells:

V8 I read in the book a little and if I can't cope, then I take it with me and ask at an exercise class. That's the most common way of coping with it. Or I ask a friend, "Have you worked this example out?" Then you can go through it together. [...] I usually try to avoid going into it too much; I don't sit with it for hours, but I come back to it another day. Maybe I'll ask somebody about it. But then I go back to it again, and then I might have had time to think it out, and when I come back to it it isn't at all so hard as it was at first

he is telling of a common experience of talking about problems, trying to get help, letting things rest and take their time to sink in. On the other hand, when V2 says:



V2 To learn mathematics, it must be mostly about understanding the question, so that you can, you know a bit of maths intuitively so it's a matter of understanding the question and using the maths you know, you bring it to a level you can handle. It might not work every time, but then you can generalise the problem and make it easier

he seems to have raised reflection—on the problem, on the ways of tackling it and on mathematics learning—to another level where it is seen as a support for generalisation. He continues:

V2 If you are going to learn something quickly you say to yourself “that’s what it says in the book”, you don’t link it to your own experience, to reality. [...] You should see the things as they really are and stop thinking of it as a mass of printed text.

Expressing a strategy of looking beyond the given problem, the text, to the meaning of the mathematics and its relation to experience, to reality.

If we return to the second strategy, “doing examples”, we can see there a watershed between the two sets of intentions and goals, which actually divides the set of categories into two parts. Doing examples in order to become proficient at that sort of example points to satisfying the demands of the task or course, as does “just learning”. Doing examples in order to understand the theory and principles, on the other hand, points to going beyond the demands of the teacher, towards striving to understand mathematics, as do the categories unstudied and studied reflection.

This is an example of a more general result, that there are two distinctly different ways of going about learning tasks: a surface approach which focuses on the “sign” of the task and a deep approach which seeks the “signified”. Every sort of task and each subject area needs to find its own description of deep and surface approaches, but here we have one for the overriding approach to learning mathematics in the first year of an engineering context. I want to say again, that this is not a characteristic of individual learners and, as we will see in more detail in due course, is closely related to how they find the situation they are learning in. A single student can adopt a surface approach in one course and deep approach in another, or a surface approach to one task in a course and a deep approach to another.

## What drives learners to do what they do?

The “why” of learning asks the question, what can we say about why students act as they do? What drives them to one approach or another in a particular situation? This is, of course, a very complex question and deserves a thorough and varied answer. But here we can consider one factor in what drives students to do what they do do when learning mathematics, and that is what they think studying mathematics is all about. A belief about the subject’s nature and its role in the world at large, whether consciously or unconsciously held, obviously drives people to act in different ways.

In the study already mentioned, students were asked in interviews such questions as, “What do you think mathematics is all about, actually?”, and “What role does

maths play in your studies”, and “Where do you think maths will come into your profession, eventually?” On the basis of discussions that followed these and other openings, three qualitatively different ways of understanding mathematics emerged (Table 2).

**Table 2.** Three categories of description of the experience of mathematics

<p><b>Mathematics as a subject of study</b> sees mathematics as a part of the degree programme, to be studied via various teaching and learning techniques</p> <p><b>Mathematics as the basis of other subjects</b>, both for study and in the world at large, sees mathematics as something existing in its own right, something to be tackled (learned or understood) for future appropriate use</p> <p><b>Mathematics as a tool for analysing problems that occur in the world</b> at large and hence solving them, sees mathematics as something which co-exists with other areas of knowledge and supports the study and development of that knowledge</p>
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A student who says the following:

I What is mathematics, do you think, generally speaking?

X2 Numbers, plus and minus, that sort of thing. I don't really know what you mean by, “what is mathematics?” Well, mathematics is a subject for me, actually, I've always seen it as a subject, something to use in other subjects

points to both the first two categories: mathematics is a subject of study and at the same time a support for other subjects of study but it has no specific role to play in analysing problems or worldly phenomena. The next quote points more emphatically to the second category:

I What is mathematics, do you think, speaking generally?

X3 A lot of stuff you have to know by heart [...] It's a tool for other sciences you might say

I Do you think you need to know it by heart?

X3 Yes, I now realise that you need to know mathematics for everything else. Now we are starting with mechanics, and it's all mathematics, but applied

A third quote, in contrast, relates mathematics and learning it to a future of problem solving rather than manipulating numbers and formulae:

I What do you see mathematics as?... How would you describe mathematics?

V7 Mmm... the ability to solve problems and... well... to meet problems and analyse them... you can reason about things in general terms with the help

of mathematics. [...] I suppose it's computers that do most of the work nowadays, but as I said, ... mathematics isn't... it isn't about that... it's about solving problems, that's the training you get [...] it's the thinking, the analysing, the reasoning...

These three quotes give a flavour of the variation which can be found in the ways a first-year class of engineering students conceive of mathematics and its place in their studies and working lives. Much more could be said on this, but let it suffice to say there is a variation which inevitably leads to a variation in ways students go about their studies.

Returning to our relational view of knowledge, it can be seen that these three categories describe relations: the relation between the learner and mathematics, the relation between the learner and her or his studies, and the relation between the learner and the secondary relation between mathematics and the phenomena in the world mathematics has the power to describe and handle.

There is a dimension of isolation: from mathematics as an isolated subject to mathematics as integrated into the programme of study and into the world it describes.

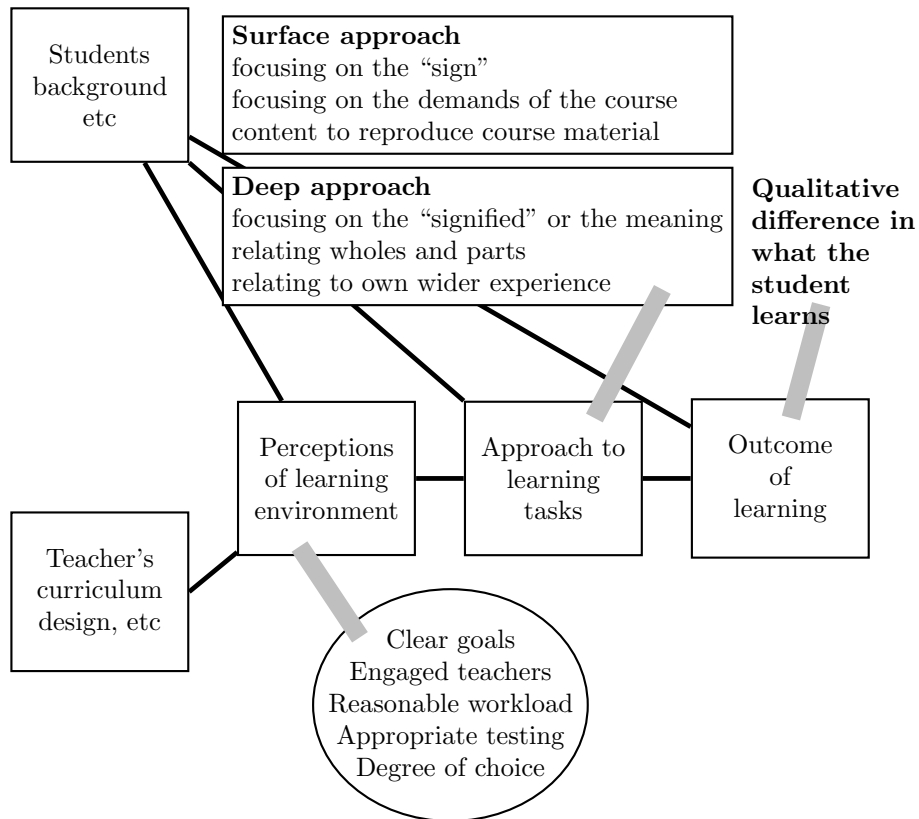
There is another dimension that is underpinned by another study which will not be described fully here but which from a different starting point reaches one set of conclusions that are rather similar (Booth & Ingerman, 2003). Students who predominantly see mathematics as a subject of study locate responsibility for learning with the teachers: the teachers know what the students need to know and they deliberately pose problems that need to be tackled by the student who will then be able to follow the track laid out by the teacher. Students, on the other hand, who have an insight and a belief that mathematics is a tool for analysing problems that occur in the world at large have also taken responsibility for learning mathematics and become autonomous as learners. This can be seen as an ethical dimension of learning (Perry, 1970), and relates to a degree of maturity as a learner and as a knower.

There is a paradox here: no-one can deny that authority lies in the end with the teacher, who does indeed know what he or she wants the students to learn, and that students are in many ways subject to this authority. But on the other hand, students have to find their own ways through the maze of knowledge they meet and make their own sense of it in an autonomous fashion. Autonomy within clear goals and guidelines is what is needed for the student to move in the right direction while retaining the right to direct themselves.

## **Teaching for understanding: A model for creating learning environments**

So far I have written of a variation in ways in which students tackle their learning of mathematics and what might lie behind and drive that. I have pointed to a qualitative difference in ways students tackle learning tasks, surface approaches focusing on trying to satisfy the demands of the task in the way the teacher appears to want it,

and deep approaches using the learning task to tackle the mathematical principles, concepts and relationships that form the content of a course.



**Figure 2.** A model of the relationships between the perceived learning environment, variation in approaches and quality of the learning outcome. After Biggs (1989) and Ramsden (1992)

When it comes to learning for understanding, the deep approach is clearly superior! Then our question becomes, how can the teacher support deep approaches among their students? Another model can help us tackle that question, a model derived on the one hand from qualitative phenomenographic research of the kind already related here, and large scale longitudinal quantitative studies that were carried out largely in the UK (Ramsden & Entwistle, 1981). The model (Figure 2) owes its origins to the work of Biggs and goes under the name of the 3-P model, the three Ps standing for presage—what comes before the learning situation—process—what happens in the learning situation—and product—the outcome of learning (Biggs, 1989).

The three central boxes in the model relate the outcome of learning to the approach to learning tasks in the way already described here, and the approaches to learning tasks to the ways in which students perceive the learning environment. It is the perception of the learning environment that drives the students to either a surface or a deep approach to the task in hand, which has been as said several times not to be a characteristic of the student but rather a response to the immediate situation.

The work of Ramsden and Entwistle identified five specific characteristics of the learning environment that affected this response. Perceiving clear goals, teachers with an interest in the students and their studies, a workload that was reasonable, that assessment practices were in accord with the form and content of the course, and a degree of freedom of choice are all associated with students adopting a deep approach to the immediate learning task. Just turn those factors round and you will see at once that the reverse is also true. Unclear goals set by teachers without apparent interest in their students but at the same time loading them with work and ignoring the form of the assessment, all in a tightly controlled course would lead the best student to a surface—let's get this over with—approach!

The links from the boxes on the far left are also results of the large-scale study: The student's history of studying, the presage, is found to affect all three of the factors while the teacher's work on curriculum and content only affects the perceptions of the learning environment. This means that the teacher's only hope for encouraging students to aim to understand their mathematics is to work on the learning environment. The content to be taught and the ways of teaching are of secondary importance, and have to take their cues from other considerations.

The paradox of autonomy is clearly related to the expression of clear goals, strengthened in the forms and content of assessment, around which the students have a large degree of freedom. Integration of mathematics into the programme of study and the world at large is less an issue of couching problems in the language and concerns of the engineering specialisation and more an issue of situating the environment for learning in the engineering specialisation and in the world. That is not to say that mathematics should be taught by the engineers—perish the thought!—but that mathematicians and engineers could unite some of their courses so that the students experienced a team of teachers leading their learning of mathematics in the world of engineering they intend to enter. Imagine a course of structural mechanics for future civil engineers in which tutorials were given by mathematicians, dealing specifically with the mathematics met in the course; or a course of field theory for electrical engineers which brought complex analysis up as a subject of study, to be taught by mathematicians in an integrated manner. And imagine access to short mathematics revision workshops with content on the web, designed and run by mathematicians, which students could refer to when they felt the need, or when advised to by teachers of engineering specialisations within, possibly, the context of large group projects.

## Designing curriculum for understanding

While the mathematics that is to be taught is very much the domain of the mathematicians, in collaboration with the engineering programme co-ordinators, the general principles of curriculum design can be considered in the light of the work presented hitherto.

We want students to understand mathematics, but what does that mean?

Depending on what philosophical standpoint one sees knowledge and learning from, understanding can have different meanings. In our relational epistemology we need to see what people at large mean by understanding, because it is that meaning that drives them. A study of children's and adults' meaning of understanding can be presented briefly, and give us a lead in our final piece in the puzzle of learning and teaching for understanding mathematics. This study results in four qualitatively different ways of understanding understanding:

To understand something is (a) to be able to do the same thing again, (b) to be able to do the same thing again in different circumstances, (c) to be able to do the same thing again in different ways and (d) to be able to do entirely different things. This explains my wording when I defined what I meant by learning: "coming to experience something in a qualitatively new and more powerful way, so that it can be accomplished in different circumstances, in different ways, and facilitate doing altogether new things".

If we relate this to understanding that the derivative of  $\sin x$  is  $\cos x$ , then the four stages of understanding, in this relatively trivial example, might be: (a) always being able to substitute  $d/dx(\sin x)$  by  $\cos x$ , (b) be able to integrate  $\cos x$ , (c) be able to relate the slope of the tangent of the sine curve to the ordinate of the cosine curve, and (d) be able to work more generally with differential equations involving trig functions. Of course, in more complex examples of understanding mathematics these stages are also more complex. It can be a nice exercise for students to take a mathematical fact and consider these four stages that mean understanding!

Our interest, and duty, is, as students can also point out, to offer them the opportunity to understand mathematics so that they can solve the problems they meet in their studies and in the real world, in an environment of clearly guided autonomy. The most important thing is to provide them with a base for coping with future situations that are entirely unfamiliar to them.

The educational researcher and developer John Bowden has used the term "capabilities" to describe this sort of understanding, and together with Ference Marton he has developed a theory of curriculum development that aims to develop capabilities in the students: the ability to cope with unforeseen issues grounded in what was learned and understood at university. This is the basis for life-long learning, in strong contrast to seeing life-long learning as an endless succession of courses to keep up to date. Naturally, courses will still be reasonable way to learn what is needed, but the development of capabilities implies that one has already determined what is needed, a result of a developed autonomous relationship with the field of work and learning.

As Bowden puts it: "University students are always learning through interaction with current knowledge so as to become capable, some years in the future, of dealing

with situations in professional, personal or social contexts that can't be specified in advance. In essence, we claim that university students are engaged in learning for an unknown future and that we have to design the curriculum with that in mind, Hence the notion of capabilities as learning goals emerges as a central idea capabilities to act in previously unseen situations.” (Bowden, in press).

He lays down 6 principles for establishing and implementing a curriculum for capabilities:

- 1 What should the learner be capable of doing at the end?
- 2 What kinds of learning experiences and in what combination would best assist the learner to achieve those outcomes?
- 3 How can the learning environment be best arranged to provide access to those optimal learning experiences?
- 4 How can the differing needs of individual students be catered for?
- 5 What specifically is the role of teachers in supporting such learning by students?
- 6 What kinds of assessment of student learning will motivate learning of the kind desired and authentically measure the levels of achievement of the intended learning outcomes?

The first and sixth of these principles form a framework for the work of teachers and teams of teachers—mathematicians and engineering specialists—who are working on developing an environment for learning for understanding. The goals of the course and the assessment must be considered together in order to form the guidance for an emerging student autonomy. Then can come the second and third into play: designing experiences—processes—within the constraints of the guidelines which can lead to the learning outcomes—products—that are implied by the goals.

When considering the fourth principle—that the differing needs of the individual student should be catered for, both the students' background—presage—has to be thought of, and the variation of ways in which students understand the role of mathematics in the programme of studies and the world—as described earlier. Of course, there are many more individual differences in a class of students, but these—together with a respect for their differences—that are of vital importance. If environments can be designed and implemented that take account of these specific differences, then the approaches adopted will tend to the deep approaches that are desirable for understanding.

The fifth principle now, finally, relates to teachers and how they can support the learning. Here I would like to suggest that teachers can best understand and support their students learning by actively engaging in research and development projects related to the practices of learning and teaching mathematics, but that is a whole other story. At least, teachers must take into account the research results on learning and understanding mathematics in all aspects of planning and practising their teaching.

There are no immediate answers on how to go about such changes to teaching for understanding, but I can summarise with a few points to draw from this paper.

Large parts of programmes must be redesigned in order to effectuate the sort of programme outlined, in collaboration between the various stakeholders—teachers of other subjects, students and ex-students—and, I suggest, educational researchers and developers. The educationalists cannot, probably, advise or provide answers on the mathematics curriculum and teaching strategies, but they can in all certainty put important questions that teachers have to answer, and they can support with the research results and approaches that are necessary to move curriculum change forwards.

The goals and its constituent parts must be articulated, which facilitate students' autonomous goal-making and support collegiality at all levels, for developing capabilities for the future which go beyond mathematics in the curriculum, and towards mathematics in the experienced world of engineering.

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# Mathematics in Engineering Education in Australia: Do We Join the Revolution?

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## Abstract

“In past years there has not been a direct relationship between the mathematics courses provided for the engineering degree programs and other (engineering) courses taught where the mathematics content could be used.” This statement was made by a project engineer responding to a survey of graduates of engineering programs seeking input on the impact of current mathematics education on their professional careers. The graduate then refers to the missed opportunity by teaching staff of linking together two second year subjects, one mathematics and one on mechanical systems. Joint teaching projects, relating mathematics to its application in engineering, is one area engineering mathematics educators are exploring to improve motivation to learn mathematics by engineering students. In Australia, there is a move by engineering schools to embrace problem based learning (PBL). What challenges does this create for mathematics staff? What are the implications for content, presentation, assessment and team work in mathematics courses in engineering education? Engineering undergraduate education is undergoing rapid change in Australia (and elsewhere). In fact there does appear to be a “revolution” taking place in engineering education. It is essential that the working relationship between mathematics departments and engineers take on a new dimension if engineering mathematics is to make a positive contribution to the revolution in engineering education or it can become the domain of engineering schools!

This paper will outline some of the challenges to be addressed and the working relationships that are taking place between mathematics staff and engineering staff to meet the challenges generated by the “revolution”.

The challenges faced by engineering mathematics educators in Australia could well be applicable to our colleagues in Europe.

## Introduction

In an Editorial for the Newsletter of the European Mathematical Society (EMS) [1] David Salinger referred to the Statutes of the EMS, in particular that one which states; “establish a sense of identity amongst European mathematicians”. Salinger argues that mathematics is international and that creating a European identity can be divisive.

From my experience mathematics in engineering education is the concern of a global village. This meeting is the result of long term cooperative interaction between Australian and European colleagues, who in many instances, face the same challenges. In fact, personally, I have gained much from my European colleagues over a period spanning 30 years. In February 2002 the Faculty of Engineering and Physical Systems at Central Queensland University commissioned a project to review the mathematics content of the undergraduate engineering degree programs offered by the Faculty. This review involved meetings with mathematicians and engineering staff at several Australian universities. These meetings, together with a survey of the current literature, and attendance at two Australian organised conferences during 2003, have given the author a reasonable insight into the “state of the art” in mathematics in engineering undergraduate education in Australia.

A review of international literature and contacts with staff in UK and Swedish universities have also given an idea of developments in Europe.

## Changes in Engineering Education—Challenges for Mathematics

A Report on a major review of engineering education in Australia was published in 1996 [2]. In the Foreword to the Report, the Chair of the Review Committee, Peter Johnson, stated that the Review recommends “*no less than a culture change in engineering education which must be more outward looking with the capability to produce graduates to lead the engineering profession in its involvement with the great social, economic, environmental and cultural challenges of our time*” (p6) “*Courses should promote environmental, economic and global awareness, problem solving ability, engagement with information technology, self learning and lifelong learning, communication, management and team-work skills, but on a sound base of mathematics and engineering science.*” (p7)

John Webster, in an overview of the Review [3] included the following reasons for it.

- The emergence of new engineering disciplines and new technologies.
- Significant changes in the capacities of tertiary students at the point of entry.

He also insisted that there should be emphasis in undergraduate programs to move away from the present focus on examinations, in one form or another. Undergraduate courses should cover problem definition and problem solving, model building and simulation.

In a paper delivered at the third Mathematical Education of Engineers conference, UK [4] the author discussed the question: Is there a gap between the changing needs of engineering education and the current service offered by mathematics departments? The paper also called for more dialogue between mathematicians and engineers by forums and special interest networks. The aim being, with joint effort, to cooperatively generate an appropriate mathematically sound but relevant and technology enhanced mathematics education for the engineer of the future.

At Central Queensland University (CQU), the Faculty of Engineering and Physical Systems has established an Engineering Mathematics Working Group (EMWG). The purpose of the EMWG, which has a membership of three engineering academics and three mathematics staff, is to generate greater awareness of the changes taking place in engineering education and the way mathematics can make a positive contribution to a changing engineering education. Groups like this one are emerging nationally resulting in greater cooperative effort to improve the mathematics education of engineers.

*The first challenge* in engineering mathematics in Australia is to generate closer, and continuing liaison between the engineering staff and the mathematicians who provide the mathematics course. Student contribution to this liaison is also considered vital. Comments, like this one from a recent graduate engineer emphasise this challenge.

“In the past years there has not been a direct relationship between the mathematics courses provided for the engineering programs and other (engineering) courses taught where the mathematics content could be used.” This statement was made by a project engineer in response to a survey of recent graduates of engineering where input was sought on the relevance of the mathematics courses to their current employment. This respondent referred to what he felt was the missed opportunity to link a mathematics course with an engineering course where the mathematics was particularly valid. Both courses were taught in isolation to each other. At a Forum on Mathematics in Problem Based Learning in Engineering Education [5] it was agreed that more time in mathematics courses should be devoted to developing understanding of concepts and less on tedious manipulation which can be efficiently carried out by relevant technology. At the same Forum, joint teaching sessions involving both mathematics and engineering staff were perceived as a means of setting the mathematics in an engineering context and so contributing to positive learning outcomes.

*The second challenge* relates to assessment and learning outcomes.

At the 2002 Australasian Association for Engineering Education (A<sup>2</sup>E<sup>2</sup>) Conference, Jackie Walkington and David Dowling presented for discussion, the paradox of best practice in assessment and the constraints of implementation [6]. They list the important principles that underpin assessment in undergraduate programs. In particular, assessment should:

- Measure student learning (and student learning is continuous).
- Be aligned with course objectives.
- Recognise student diversity.
- Encourage student learning.
- Have standards which are transparent to the students.

Whilst academic staff may well be aware of these features of sound assessment, Walkington and Dowling outline some of the barriers to implementation of these features. The student body is no longer homogeneous. There is a diversity of learning styles,

backgrounds and attitudes. Their message was that catering for this diversity requires change not only at the classroom level but at all levels within the engineering education framework. The barrier to effective assessment is the difficulty/inability to truly get to know about the diversity of the students' backgrounds. They also discuss the lack of resources and the attitude of some staff to the acceptance of changing assessment practices. In addressing the problem they argue that; (i) assessment needs to be part of the initial design of a course, both formative and summative assessment is required, and, (ii) a holistic approach is required if an understanding that effective change seeking long-term improvement to teaching/learning within a faculty is to be achieved.

In the restructured first year engineering mathematics courses at CQU (Engineering Foundation Mathematics and Engineering Mathematics) assessment now consists of assignments, projects and class tests with the component for formal examination reduced from 80% to 50% of the total.

There is also a trend in assessment for students to be allowed to have graphics calculators with computer algebra systems (CAS) in formal examinations. This trend has in itself generated a critical review of the structure of formal examinations in assessment. Associated with the challenge of linking assessment with well defined learning outcomes is the need for engineering mathematics to be perceived as being an integral component of the engineering program, not an attachment, and to be set within the context of engineering, or related applications.

*The third challenge* is to develop the process of mathematical modelling as a principal learning outcome.

Whether mathematical modelling, as a skill, art or craft involving problem identification and the process of listing assumptions, defining variables and setting up mathematical relationships, should be included in the engineering mathematics curriculum for undergraduates, continues to be a topic of debate. As well there are times when the use of existing mathematical models and the process of setting up a model has been confusing for the student.

Mathematical modelling in high school mathematics is currently receiving a lot of positive attention. The mathematics syllabuses in high schools in several Australian States now include mathematical modelling as an essential component. The Objectives of the Queensland Senior Mathematics B Syllabus (the normal mathematics prerequisite for engineering undergraduate programs) include the heading, Modelling and problem solving. Under this heading, the Syllabus states, *“By conclusion of the course students should be able to demonstrate the category of modelling and problem solving through*

- *Understanding that a mathematical model is a mathematics representation of a situation*
- *Identifying the assumptions and variables of a simple mathematical model of a situation*
- *Forming a mathematical model of a life-related situation*
- *Deriving results from consideration of the mathematical model chosen for a particular situation*

- *Interpreting results from the mathematical model in terms of the given situation*
- *Exploring the strengths and limitations of a mathematical model”*

However skills audits of the new intake of students into engineering programs at CQU reveal little understanding of this definition of mathematical modelling. It is suggested that the process of mathematical modelling (as defined in the Queensland Syllabus) can be developed in first and second year engineering mathematics. Joint teaching with engineering staff in third and fourth year could then see the process strengthened as students encounter mathematical modelling within the context of engineering problems.

Mathematical modelling at all levels of mathematics education has been the theme for the international conferences on the Teaching of Mathematics and Applications (ITCMA). At ICTMA9 the author [7] summarised the benefits to learning provided by the inclusion of mathematical modelling in the curriculum, especially for engineering students. *“If interest and motivation are developed through relevance, and students have the resources to experiment, explore and investigate the analytic, graphical and numerical aspects of a problem, content will be consolidated and new learning can take place through modelling activities”* (p146).

Richard West describes mathematical modelling as a strengthening thread in mathematics courses at the United States Military Academic, West Point [10]. He stresses that curriculum reform in mathematics should have a primary focus of empowering students and that mathematical modelling can play a vital role in contributing to a cultural change in mathematics education. West lists interdisciplinary projects, which can support this cultural change; the Interdisciplinary Lively Applications Projects (ILAP’s) (Available on the COMAP website <http://www.projectintermath.org/products/listing/>)

There is little doubt in the minds of many engineering educators that there is a definite role for mathematical modelling and it should be integrated into the curriculum.

An additional argument for modelling is that it presents a marvellous opportunity to create an environment for learning to be embedded in reality. However to expect students to formulate mathematical models when their competence in mathematics may well require strengthening can often lead to frustration and loss of motivation. It is therefore suggested that the development of the modelling process be gradual and guided but be a definite framework for engineering mathematics. To enable students to make progress with the process of modelling it is vital that they have a thorough knowledge, not just a 50% pass rate, of the foundation concepts developed in first year courses.

In the final year of an undergraduate program it is envisaged that mathematical modelling would include team projects involving mathematics and engineering staff sharing the presentation.

Setting the scene for mathematics to contribute to PBL in engineering education is *the fourth challenge*.

The momentum for engineering education in Australia to embrace problem based learning (PBL) has increased since Don Woods [8] first used the term in 1994. A

special Network link on the Australasian Association for Engineering Education (AAEE) ([www.aaee.com.au/network/index.htm](http://www.aaee.com.au/network/index.htm)) refers.

A special issue of the International Journal of Engineering Education [9] was devoted to PBL in engineering education. Questions raised in the Editorial to this special issue include: “How much PBL is expedient for a degree course, how does PBL fit into our established concepts of academic studies, and how does PBL effect the performance of brighter and more average students”. The second of these questions is pertinent to the challenge of integrating mathematics into PBL in engineering education. This is a challenge for both mathematicians who teach engineering mathematics and engineers who teach in the PBL mode. De Graaff and Kolmos [10] discuss the curriculum structure in which the subject (course) disciplines are integrated through case studies, the learning process is facilitated by the lecturer and assessment must be compatible with the learning objectives. The emphasis is testing for competence in applying the course matter rather than testing for factual knowledge.

In this special edition there is a serious omission! No paper addresses the challenge of integrating mathematics into PBL in engineering education. However, Bowe, et. al., of the Dublin Institute of Technology describe how PBL can be used to teach physics to engineering students [11]. These authors warn that the students do require a sound body of mathematical skills and that facilitation of the learning is a key feature and tutors must be aware that students are only in the early stages of self directed learning.

Using PBL processes to teach physics is, possibly, similar to teaching engineering disciplines by PBL. However in the case of mathematics, the main challenge is that the students need a deep understanding of the first year concepts. If an engineering problem requires a knowledge of eigenvalues, for example, this topic cannot be fed into the PBL project without the student having a sound prior knowledge of matrix algebra. There is no doubt that addressing the task of how mathematics can contribute to PBL in engineering education is a major challenge.

*The fifth challenge*, and by no means the final one, is how mathematics staff can make a positive contribution to the move for mathematics to be presented as part of an integrated curriculum. As outlined, mathematical modelling, especially in the advanced courses, and PBL in engineering education become multidiscipline in nature.

Integrated curricula are being developed.

An example is the Principia Program at the Institute of Advanced Studies in Technology in Mexico. This excellent example of an integrated teaching model is described in a joint paper by the Dean of the Engineering school and the Chair of the Mathematics Department [12]. An integrated curriculum of this type would make a positive step to address the challenge of providing a mathematics service for PBL engineering education. As Daniel Goleman points out in his latest book [13] “There is a crucial difference between *declarative* knowledge, knowing a concept and its technical details, and *procedural* knowledge, being able to put those concepts and details into action” (p242). It is the blend of declaration and procedural knowledge resulting from participation in engineering mathematics which can be achieved by

cooperation and sustained effort by engineers and mathematicians which will determine the future blend of mathematics engineering and education in the education of Australian engineers of the future.

## Conclusion

In 2002 the Mathematics Working Group (MWG) of the European Society for Engineering Education (SEFI) presented a core mathematics curriculum for the European engineer [14]. Apart from listing detailed content of topics in mathematics, the curriculum also draws attention to the on-going challenges presented by

- The diversity of mathematical ability of entrants to engineering programs.
- The need for those teaching mathematics to be aware of applications of mathematics in engineering and changes taking place in mathematics education in high schools.
- The early introduction of mathematical modelling into the education of engineers.
- “Traditional” methods of assessment—do they really meet the objectives of learning outcomes?
- The role of technology in mathematics for engineers the concern of the “black box approach” favoured by some engineers.

So the challenges are also being addressed by this Group. Looks like we all join the revolution!

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# Assessment in Engineering Mathematics

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## Abstract

In this paper we describe the background to and preliminary findings of the *SEFI* Mathematics Working Group (*MWG*) assessment project. The paper then goes on to address some emerging issues in assessment such as the use of constructive alignment in curriculum development and the linking of assessment tasks to intended learning outcomes.

## 1 The *SEFI MWG* Assessment Project

The *SEFI MWG* assessment project has three main purposes:

1. To survey the methods of assessment used in engineering mathematics across Europe;
2. To stimulate debate on appropriate and efficient methods of assessment;
3. To spread good practice in assessment.

This is an on-going project which is currently at a relatively early stage. The findings reported here should be taken as no more than an interim report.

A questionnaire on a number of issues related to assessment in engineering mathematics was sent to all contacts on the *SEFI MWG* database (a copy of the questionnaire is given in Appendix A). Further copies of the questionnaire were distributed at the 12th *SEFI MWG* seminar in Vienna in June 2004. In addition, there was a round-table discussion at the seminar on issues relating to assessment.

## 2 Findings of the Questionnaire

By the time of the *SEFI MWG* seminar in Vienna responses had been received from colleagues across Europe; from eleven different countries. The full list of countries from which responses were received is given in Appendix B. Typically only 2 or 3 responses were received from each country. Further questionnaires were completed by delegates at the 12th *SEFI MWG* seminar and the questionnaire will be distributed again at a number of national conferences in the near future.

As can be seen in Appendix A, respondents were given a list of methods of assessment and asked to grade each method for both themselves and their institution on a three point scale

- Not used at all
- Used occasionally
- Used frequently

The responses indicated that there is considerable variation in practice across Europe. Several respondents indicated that their own institution used only one or two different methods of assessment, whilst others showed that a wide variety of assessment methods are in use at some institutions.

Unsurprisingly, written examinations are the most widely used form of assessment, with the ‘closed book’ form being much more widely used than ‘open book’. Several respondents reported the use of examinations where the candidates had access to computer facilities in order to help them answer the examination questions.

Oral examinations are used quite widely, particularly in central European institutions. In some institutions they are used frequently and one respondent used the free comment section of the questionnaire to state that oral examinations are the most important form of assessment because they give the best opportunity to test in-depth understanding of material. Others commented that this kind of assessment is highly staff intensive.

Take away assignments are used at several institutions, but always as one amongst a number of methods of assessment and never as the only or primary method. From remarks in the free comment section, it would appear that some staff have reservations about this method of assessment because it is impossible to be certain that the student submitting the work actually did it for him/herself. However, take away assignments were seen as giving students an opportunity to explore more realistic problems than they can in an examination and for this reason often require the use of computer software to complete the assessment task.

Only a few institutions use multiple choice tests—and those that do use them do so only occasionally. Such tests can be cheap to administer as they can be computer delivered and marked and so they can be useful in giving formative feedback. However, as all that is marked is the student’s final answer, they have limitations when being used for summative assessment.

Other methods of assessment such as project work, group work and oral presentations are not widely used.

### 3 Constructive Alignment

Constructive alignment is an approach to curriculum design which requires explicit integration of the intended learning outcomes, teaching methods and assessment in order to produce efficient student learning. In Britain, the Quality Assurance Agency has been a leading proponent of this approach [1]. This approach is frequently implemented in a tight hierarchical structure.

The top level of the hierarchy is the programme (or course). A Programme Specification must be published which states explicitly the intended learning outcomes of the programme. A programme will be made up of modules or units each with

their own intended learning outcomes. The programme learning outcomes should be demonstrably delivered by a combination of module learning outcomes.

Traditionally modules/units were described by a syllabus that would be covered. This was often a list of topics and gave no indication of the depth of coverage of these topics nor of what would be expected of students. So, for example, in history the topic ‘First World War’ would appear on syllabuses from primary school to post-graduate course. Intended learning outcomes give more detail than syllabus topics. They specify depth and what students should be able to do at the end of the module. Typical intended learning outcome statements begin ‘On successful completion of this module students will be able to’. What follows has a taxonomy of its own, but a key point is that the verb which follows the introductory statement should be something which can be assessed.

To a limited extent the hierarchical nature of mathematics reduces the need for detailed intended learning outcomes, but they are nonetheless used. So, for example, rather than a syllabus list which gives a number of topics in calculus, many modules in engineering mathematics contain statements such as ‘On successful completion of this module the student should be able to apply standard techniques in algebra and calculus to engineering problems’. There is still a need for a syllabus or topic list to contextualise what is meant by ‘standard techniques’. The current version of the *SEFI* Core Curriculum for Engineering Mathematics [2] is written in learning outcomes form.

The next stage of the hierarchy is in terms of assessment. The intended learning outcomes of the programmes have to be assessed. However, assessment usually takes place at module level. This is not a problem as programme learning outcomes are delivered by a combination of module learning outcomes. It is therefore important that module learning outcomes are assessed. In programme validation, it is now common to expect explicit links between module assessment activities and the intended learning outcomes.

The assessment questionnaire included questions relating to learning outcomes and whether or not there was any explicit link between these and assessment methods. The replies to these questions are summarised in Table 1 below.

<b>Use of Learning Outcomes</b>	<b>%</b>
Do not use learning outcomes	22
Use learning outcomes but no link to assessment	28
Use learning outcomes and link to assessment	50

**Table 1:** Use of Learning Outcomes

## 4 Conclusion

The preliminary findings of the *SEFI MWG* Assessment Project indicate that there are many different approaches to assessment used in engineering mathematics across Europe. Whilst traditional methods such as written and oral examinations are still

most common, there is increasing use of take away assignments and assessments which require students to use computer software.

In curriculum development, learning outcomes are now widely used throughout Europe. Furthermore, it is becoming increasingly common to link explicitly the learning outcomes and the assessment regime.

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## Appendix A—The Questionnaire

### SEFI Mathematics Working Group Engineering Mathematics Assessment Questionnaire

Assessment of Engineering Mathematics is currently a topic of considerable interest across Europe. This is one of the themes of the 12th SEFI Mathematics Working Group Seminar to be held in Vienna in June 2004.

As part of this seminar, the Organising Committee would like to be able to present an overview of practice in assessing Engineering Mathematics throughout Europe. In order to be able to do this we would be grateful if you would complete the questionnaire below.

When you have completed this questionnaire, please return it as an email attachment to Prof Duncan Lawson, email [d.lawson@coventry.ac.uk](mailto:d.lawson@coventry.ac.uk)

#### General Details

Name: .....

Country: .....

Email address: .....

#### Methods of Assessment

In the questions below we ask you to give some information about the different forms of assessment that are used by you and by your organisation in the assessment of Engineering Mathematics. By ‘your organisation’ we mean either your university, your faculty or your department—whichever you are able to answer for.

Name the organisation to which the answers refer:

.....

Number of engineering first year students entering the organisation: .....

Do you specify ‘learning outcomes’ that students should achieve in engineering mathematics? **YES / NO** \*

If yes, do you explicitly link assessment tasks to learning outcomes?

**YES / NO** \*

\*(please delete as appropriate)

Please note the following definitions regarding the questions below.

Closed book exam: an exam where students cannot take any material into the exam

Open book exam: an exam where students can take materials—such as revision notes or text books—into the exam with them.

Please complete the table below, answering separately for yourself and for your organisation, using the following scale:

- 0 - not used at all**
- 1 - used occasionally**
- 2 - used frequently**

Please leave boxes blank where there is no answer (for example, if you do not teach engineering mathematics beyond the first year).

Assessment Method	Used by you		Used by org	
	1st year	Later years	1st year	Later years
Closed book written examination				
Open book written examination				
Closed book examination with computer facilities available				
Open book examination with computer facilities available				
Take away written assignments				
Take away assignments requiring the use of computer software				
Oral examinations				
Computer-delivered assessments				
Multiple choice tests				
Project Work				
Group Work				
Oral presentations				
Other method #1				
Other method #2				

If you use any other methods please describe them below.

**Good Practice**

Please outline any assessment practice that you use, or would like to use, that you feel is particularly innovative or represents good practice in assessing engineering mathematics.

## **Appendix B—Countries from where completed questionnaires were received**

Questionnaires were returned by colleagues in the following countries:

- Czech Republic
- Finland
- France
- Germany
- Hungary
- Latvia
- Lithuania
- Norway
- Slovenia
- Sweden
- United Kingdom



# Contributed Papers

# Visualization, Conceptual and Procedural Knowledge in Mathematics for Engineers. The Case of the Definite Integral

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## Introduction

Applications of mathematics to engineering require the use of basic concepts in contexts of different degrees of generality and complexity. The teaching of a certain topic or subject does not necessarily imply, as a result, the understanding of such a topic or subject by the student (in the sense of understanding a mathematical concept given by Sierpiska, 1992). Access to advanced mathematical concepts requires the interaction of processes of representation, abstraction, modelling, generalization and synthesis.

Within the complexity of these processes, visualization plays a crucial role in the learning of mathematics, because of the interpretational power and synthetic vision involved in this way of dealing with information. The capability of coding and decoding different levels of visual information provides a resource to assess students' conceptualisation.

As a result of instruction that has a strong tendency to formalism, on the one hand, and a focus on the development of complex procedures, on the other, previous studies have revealed that our students visual reasoning (Cavallaro & Anaya, 2003). This fact may place low value on contribute to keeping hidden the underlying mathematical ideas.

In the case of the definite integral, in order to pass the final exam, our students must be able to solve problems like:

- A. A function  $f$  is such that  $e^{f(x)}f'(x) - x = 0$ . The graph of  $f$  contains the point  $(0, 0)$ . Calculate the area bounded by the graph of  $f$  and the lines  $y = 0$ ,  $x = -2$ ,  $x = 2$ .
- B. If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and has a minimum at  $x = 0$ , find the Taylor polynomial of 2<sup>nd</sup> degree at  $x_0 = 1$  associated to the function

$$F(x) = \int_0^{x^2-1} f(t) dt.$$

However, being able to develop the necessary procedures to solve these sorts of problems does not imply an adequate comprehension and conceptualisation of the definite integral concept. Sometimes, students apply procedures without understanding what they are doing: for instance, in their own words, they explain that “*to differentiate this integral, we have to write the function that is inside, but evaluated at the upper limit of the integral, and multiply it by the derivative of the limit. I am not sure why. I think that there is a theorem that states it*”.

In this study, we have investigated what are the conceptions that students have about integral, area and integral function after they have passed the course. The ability to code and decode visual representations allowed us to analyse these conceptions in connection with procedural and conceptual knowledge and to detect misconceptions that were not revealed by standard evaluations.

Insufficient treatment of graphical-visual tasks together with a weak valuation of visual reasoning could be one of the sources of students’ difficulties in the conceptualisation of a definite integral.

## Theoretical considerations

Visualization in mathematics is not a mere immediate vision of relationships among mathematical concepts, it is an interpretation, a real work of coding and decoding information. It is closely related to a process of communication of ideas and is deeply rooted in the long history of the mathematical activity (Guzmán, 1996).

Visual intuition in mathematics has led to ideas that have resulted in great advances in the development of mathematics. However, a person will be able to take advantage efficiently and effectively of visualization as a resource only if (s)he learns how to understand adequately the meaning of what is communicated. The more complex the codification of ideas is, the more conceptual is the mathematical knowledge involved in the process.

Historically, researchers in mathematical education have considered the existence of two kinds of mathematical knowledge, which have taken different forms: skill vs. understanding; meaningful vs. mechanic. Skemp (1978) distinguished between *relational and instrumental understanding*.

Hiebert and Lefevre (1986) propose a classification in conceptual and procedural knowledge.

*Conceptual knowledge:* It is the knowledge characterized by being rich in relationships among units of information. There are two levels at which these relationships can be established:

- a) Primary level: the relationship is constructed at the same level of abstractness as that at which the information is represented (the term abstractness refers to the degree to which the relationship is tied to specific contexts).
- b) Reflective level: the relationship is constructed at a higher level of abstraction than the information represented, transcending the level of the represented knowledge.

*Procedural knowledge:* It is characterized by its structure: procedures are hierarchically arranged so that some sub-procedures are embedded in others forming a linear sequence of prescriptions. Procedural knowledge encompasses two kinds of information:

- a) Formal language or symbolic representation system in mathematics.
- b) Rules or algorithms for solving mathematical tasks.

Students are not completely competent in mathematics if one of these, conceptual and procedural, knowledge is deficient or if having been acquired, they remain as separate entities.

## Method

A questionnaire was presented to 72 students of the National University of Technology, some months after they passed the final exam of the first course of Mathematical Analysis.

The questionnaire stressed different levels of difficulty in the coding and decoding visual information about areas, primitives, Riemann integral and integral function.

A qualitative analysis of the data was done, through the study of the justifications and the type of resolutions students presented, considering whether it was founded in a visual analysis.

## Some of the Questions, Relevant Results and Discussion

Some questions, in which a rather low level of visualization was required, were aimed at studying the discrimination between area and definite integral

Q1. Calculate the area of the region bounded by the graph of  $f(x) = x^2 - 3x$  and the  $x$  axis in the interval  $[0; 6]$

Q2. Calculate

$$\int_0^3 (x^2 - x) dx.$$

Interpret the result graphically.

74% of the students answered the first question incorrectly, and half of them, didn't draw any graph. 56% calculated the integral instead the required area. The correct answers were accompanied by a graph.

In Q2, the required calculation is direct, regulated only by rules and procedures, and was answered correctly by the 65% of the students. However, only 17% interpreted the result correctly. Among those who attempted the required graphic interpretation there were a variety of contradictory responses like: *"It is not the*

area” or “the result of the integral is equal to the area”. These students didn’t visualize the integral of a negative function in  $[a,b]$  as “(-) the area of the region determined by  $f$  with the  $x$  axis in  $[a,b]$ ”. In this case, concepts and procedures were not connected.

Other questions explored about the same issues, but demanding from students a higher level of codification, requiring the construction of examples, like these ones:

Q3. Show the graph of a non negative function in  $[a, b]$  but with null integral in this interval.

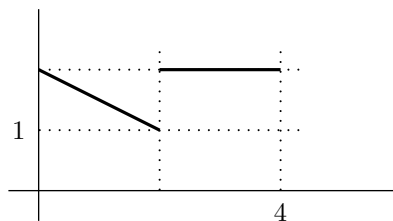
Q4. Show the graph of a function in  $[a, b]$ , such that the definite integral is smaller than the area bounded by the curve in the interval.

In Q3, 49% of students responded incorrectly. 31% of students did not pay attention to the requirement of “ $f$  nonnegative”. This situation reveals a fact that had been already observed in other studies related to visualization (Cavallaro & Anaya, 2003): many of the students cannot attend to all the requirements, focusing a just one of them.

In Q4 only 52% of the students answered correctly, showing again a gap between the concept of area and definite integral. The comparison with the results of Q1 (only 17% of correct responses) seems to indicate that when the function and the interval is given, the students focus mainly on the procedure of calculation, neglecting the meaning.

Other questions were aimed to identify Riemann integrable functions on an interval, like:

Q5. (Labraña, 2000)  $f$  is defined in  $[0, 4]$ . The graph of  $f$  is what is shown in the figure.



- a) Is  $f$  integrable in  $[0, 4]$ ?
- b) Is it possible to calculate the area of the region bounded by the graph of  $f$  and the  $x$  axis in this interval? Explain your answer.

Q6. Show the graph of a discontinuous but integrable function.

The responses to these sort of questions revealed a dissociation between the concept of area and Riemann integral and errors about the concept of integrability. In Q6, the construction of a graph is required. There were 87% of correct responses, though only the 48% considered a discontinuous function with finite jump, 17% of students believed that an integrable function may have at most a removable

discontinuity, though they used integrals to calculate the area. This fact shows that the continuity of a function will not persuade or dissuade a student to connect or not integrals and areas, they will do it for calculus purposes even if they are convinced that the function is not integrable. Visualization of an integral as an area seems to be connected to procedures for calculating, rather than to a concept.

In Q5 the sufficient conditions for existence were mistaken for the necessary ones, what was evidenced in answers like: “yes, because  $f$  is bounded” or “no, because  $f$  is not continuous” (what is consistent with other researchers’ findings, as Labraña, 2000). The incorrect responses could be the result of a lack of variety in graph and analytic examples of discontinuous functions.

Some other questions were aimed at studying the relationship that students establish between the graph representation of the primitives of a given function on an interval. One of them (Q7), presented the graph of a primitive of  $f$ , and required drawing another primitive of the same function.

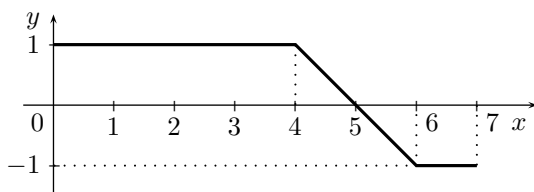
Though 59% of the students responded correctly, the majority of incorrect answers were based on incorrect visualizations of the primitives, like horizontal shift of the curve, or reflections with respect to the  $x$  axis, mistaking “another primitive” for a function that bounded the same area.

Though these questions presented a direct relationship between concepts and visual interpretations and thus, a rather simple process of codification and decodification of visual information, the students show a gap between procedural and conceptual knowledge, which became evident during this process.

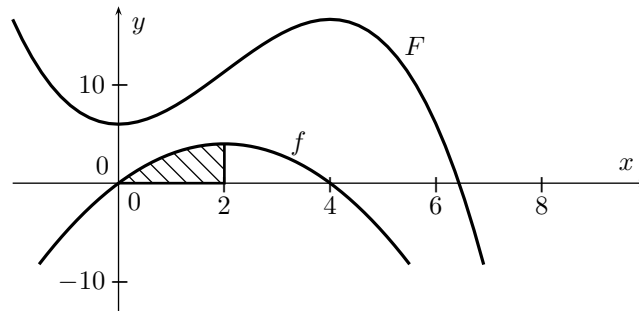
Other questions, required a higher ability in coding and decoding visual information and a higher level of abstraction and conceptualisation of the involved notions.

Some of these questions inquire about the connections between definite integral, integral function and area, for example:

- Q8. The figure represents the graph of a function  $f: [0, 7] \rightarrow \mathbb{R}$ . Let us consider  $F: [0, 7] \rightarrow \mathbb{R}$  the function defined by  $F(x) = \int_0^x f(t) dt$ . Calculate  $F(4)$  and  $F(7)$ . Draw the graph of  $F$ .



- Q9 Let  $F$  be a primitive of  $f$  in  $[0, 2]$ . Is there any relationship between the shaded area and the length of the segment marked on the  $y$  axis? Explain your answer.



These sort of questions require reverse thinking,—a kind of reflective abstraction (Dubinsky, 1992)—and a process of synthesis—that intervenes in the process of understanding mathematics—to perceive the relations between facts apparently dissociated (length of the segment and area of the region), organizing them in a consistent whole. (Sierpiska, 1992.)

In Q8 only 29% of the students were able to obtain the required results, and most of them visualized it from the graph. The others, failed in an attempt for analytic solution. Some of them mistook  $F$  for  $f$  or showed inconsistency between the graph and the analytic solution.

Only 18% of the students were able to draw the integral function and most of them weren't able to visualize the way to integrate a piece-wise function when the upper limit of the interval is variable.

In Q9 the percentage of non-respondent students was very high: 65%, and only 23% of the answers were correct.

The difficulty of this problem lies in the requirement to relate magnitudes visually associated to different conceptual systems: length of the segment = ordinate of  $F(x)$  and the area bounded by the graph of the function  $f$ . Understanding the concept of integral function requires that students may be able to visualize the “accumulated integral ‘up to  $x$ ’”. This concept is cognitively different to the notion of a primitive as an anti-derivative, and it is precisely, the fundamental theorem of calculus that integrates these notions in the same schema. Students should have a correct mental schema of this theorem to answer this question. The conceptual knowledge involved in it is constructed at a higher level of abstraction than the information that it is connecting (definite integral and derivative): it is a knowledge constructed at a reflective level.

## Final remarks and recommendations for teaching

The capability of coding and decoding different levels of visual information provided a good frame to assess students' conceptualisation about the involved notions.

A general conclusion of this study could be that insufficient training in handling visual information together with a poor valuation of visual resources could possibly affect students conceptualization and performance in several tasks related to the definite integral. However, it would be sensible to develop deeper studies in order to confirm this assertion.

At the procedural level it was observed that:

- The students showed a strong preference in applying rules and procedures and not always were able to use graphs to guide their reasoning and resolution.
- The definite integral turned out to be for them a mere juxtaposition between the anti-derivative and the Barrow rule, of the type “rules without reasons”.
- The students were not able to cope with all the requirements or constraints of a problem at the same time, neglecting important information.
- Drawing graphs or interpreting them was a difficult task for these students. Difficulties in the correct visualization of the resulting 2D-regions may become a source of errors.

In the conceptual level it was observed that:

- The students’ responses revealed dissociation between the concept of an area and Riemann integral.
- The tasks involving continuous or discontinuous piece-wise functions showed that the students have not assimilated the concept of integral function which is knowledge to be constructed at a reflective level.
- Integrating from the graph implies the connection of concepts at a higher level of abstraction and reverse thinking. The students were not able to respond adequately to these requirements.

The deficiencies that these students presented when dealing with visual resources might be caused, on one hand, by the excessive emphasis that teachers put on analytic resolutions, and the rejection of visual reasoning, considering it intuitive and inaccurate, and, on the other hand, by the students’ tendency to compartmentalize conceptual and procedural knowledge related to the concept of the integral.

Training students in activities related to codification and de-codification of complex visual information may help them:

- ◊ To develop the processes of representation, abstraction, modelling, generalization and synthesis and a reversibility of thinking, which are important components of advanced mathematical thinking.
- ◊ To get a holistic view of each problem-situation

This could be done in two ways:

1. By encouraging interpretation of graphs, and of numerical data when solving a problem, and relating them to possible images (applied mathematicians do this).
2. By encouraging the construction of examples and of graph of typical situations and of non-standard situations, in order to elicit synthesis and abstraction processes and also to control misleading visualizations and intuitions.



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# The SEFI-MWG Core Curriculum and its Application to Hierarchical Syllabus Design

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In 2002 the SEFI Mathematics Working Group revised its original 1992 advisory curriculum, [1] and replaced it with Mathematics for the European Engineer—a Curriculum for the Twenty-first Century, [2]. This is a more structured and detailed document than the original and includes a phrasing of the curriculum in terms of learning outcomes rather than a list of topics to be covered. The increased detail is finely itemised, and Levels 0 (school), 1, 2, & 3 of core mathematics specified with general attention to the various disciplines of engineering.

The SEFI Core Curriculum, [2], or SEFI CC, serves as an advisory document to those designing syllabuses in mathematics for engineers in universities across Europe. The Bologna Declaration calls for a recognition between syllabuses with a view to student exchange between institutions and countries so designers will seek a mechanism by which these can be contrasted in detail. The SEFI CC provides an ideal resource to input as raw material. Each of its three lower levels (0, 1, 2) is divided into four or five component parts in which lie the main topics and subtopics. Levels 0, 1, 2 represent hierarchical progression from school forward to the first two stages of university education. Also, the subtopics within each topic are largely progressive as well, whereas the main topics comprise a useful sequence of delivery, which is suitable to the teaching of a course. The hierarchy is thus broadly in place and the work currently under way is aimed at structuring the subtopics within the topics into clusters of learning objectives. Meeting the learning objective within a cluster rests upon a learning achievement in preceding clusters, topics and levels so

the Curriculum is being put into the form of a directed graph comprising aspects of a tree structure.

Before we give a practical example of how the directed graph might operate in a particular case we need to remind ourselves of the overall schemata of the SEFI CC.

Core Zero

Algebra	(0A/pre-core)
Analysis and Calculus	(0C)
Discrete Mathematics	(0D)
Geometry & Trigonometry	(0G)
Statistics & Probability	(0P)

Core Level 1

Analysis and Calculus	(1AC)
Discrete Mathematics	(1DM)
Geometry	(1G)
Linear Algebra	(1LA)
Statistics & Probability	(1P)

Level 2

Analysis and Calculus	(2AC)
Discrete Mathematics	(2DM)
Geometry	(2G)
Linear Algebra	(2LA)
Statistics & Probability	(2P)

Level 3

[Elective Topics—based upon the above]

Core Zero starts off with Algebra as its underpinning component part. This includes the most basic but relevant highschool pure mathematics. It comprises 4 main topics, arithmetic of real numbers; algebraic expressions and formulae; linear laws; quadratics, cubics and polynomials in general. Within the topics are 14/17 subtopics making 60 subtopics overall. By subdividing each of the topics into three to five identifiable clusters we can choose a subtopic, e.g. sketch the graph of a quadratic equation, and argue that the learning objective behind it demands much of the knowledge of linear laws such as Cartesian co-ordinates and straight line graphs, but possibly not including linear inequalities. The knowledge behind linear laws in this respect rests upon definable elements of arithmetic and algebraic expression. And this is just the beginning, as ‘0A’ underpins almost all of ‘0C’ and key elements of ‘0G’ whereas the very small components ‘0D’ and ‘0P’ have a more empirical startpoint and need new concepts and a much reduced dependency on ‘0A’.

The ‘pyramid of knowledge’ principle described so far can be extended throughout the entire SEFI Core Curriculum in layered cake form. The Levels 0/1/2 assist considerably with general definitions though protocols need to be adopted in defining terminologies acceptable to syllabus designers. At each level it is best to take analysis and calculus first of all as this very largely subsumes its utilisation in the other component parts, for example ‘2P’ might use ‘2AC’. Very occasionally there is a parallel interrelation between component parts, usually geometry and linear algebra. Also an acknowledged progression in the subtopics within a topic is necessary so

that successive clusters are progressive in that topic. Sometime though the rule can be broken. At a deeper level of refinement, one can reasonably argue, for example that a student could apply the method of integration by parts to indefinite/definite integrals by performing such on polynomial or power functions but were it needed to solve practical problems which require the evaluation of an integral then rational, trigonometric, exponential or logarithmic functions should likely be included. This represents a much higher level of drill and practice and this would always be true whenever practical problems are involved. Furthermore any use numerical methods, software, or computer algebra practice demands a considerable knowledge of the underlying analytical principles and limitations. This is reflected in the structure being proposed.

It is planned to store the full range of topics and subtopics in an interrogative database. By choosing a subtopic we identify the cluster and include any subtopics within it as necessary. We also identify the immediately supporting clusters/topic (and the subtopics within them), and recursively through all others down to the bottom of Core Zero.

A discussion group at the 12th SEFI-MWG Seminar is invited to consider the structuring of the SEFI CC in this form. The hope is that the database will be developed at the University of Bristol and will be made available to the academic community.

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### **Annex A:** Illustration of the Track of Hierarchical Dependency

#### **‘Understand how existence and uniqueness relate to the solution—of an ordinary differential equation’**

##### **2AC** Analysis and Calculus

###### Topic 1—Ordinary Differential Equations

###### Cluster 1

Understand how rates of change can be modeled using 1st & 2nd derivatives

Recognize the kinds of boundary conditions which apply in particular situations

Distinguish between boundary/initial conditions

Distinguish between the general/particular solution

Understand how existence and uniqueness relate to the solution

Classify differential equations and recognize the nature of their general solution

*Rests on:*

**1AC** Analysis and Calculus

Topic 5 Differentiation

Cluster 1

Understand concepts of differentiation and smoothness

Differentiate inverse functions

Differentiate functions defined implicitly

Differentiate functions defined parametrically

Cluster 2

Locate any points of inflection of a function

Find greatest/least values of a physical quantity

Topic 7 Methods of Integration

Cluster 1

Obtain definite, indefinite integrals of rational functions in partial fraction form

Apply the method of integration by parts to indefinite/definite integral

Use the method of substitution on indefinite/definite integrals

*Rests on:*

**0G** Geometry and Trigonometry

Topic 5 trigonometric identities

Topic 4 trigonometric functions

Topic 3 Co-ordinate geometry—not polars

Topic 2 Basic trigonometry

Topic 1 Geometry

**0C** Analysis and Calculus

Topic 6 indefinite integration

Topic 5 stationary points

Topic 4 rates of change/differentiation

Topic 3 logarithmic/exponential function

Topic 2 sequences/series/binomial expansions

Topic 1 function and inverses

*Rests on:*

**0A** Algebra (all)

# Intuitive Reasoning in Early and Advanced Mathematics for Engineers

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## Introduction

The formation of engineers requires the development of a logical mathematical thinking, however, it is well known that the acquisition of this sort of thinking is a slow and difficult process. As a consequence, there is a need for systematic studies of the specific cognitive difficulties of engineering students who deal with quite advanced mathematical concepts. Within a research project in mathematics education developed in the National University of Technology, several research works were conducted with the aims of detecting and explaining the above mentioned difficulties and of finding theoretical bases to recommend remedial actions in teaching practice.

Many of the difficulties we are working on are related to intuitive mental models, intuitive heuristics and their interaction with logical schemas and formal knowledge. In this presentation, the results of an exploratory study composed of three research works will be shown. These studies were aimed at identifying the intuitive models, heuristic and intuitive reasoning and its relation with schemas and formal knowledge in groups of engineering students in the first, second and third year of mathematical courses.

A common framework to study the intuitive processes is the theory provided by Fischbein (1987). According to Fischbein, an intuition is a cognition that appears subjectively as self-evident, directly acceptable, holistic, coercive and extrapolable. Coerciveness and self-evidence of intuitive ideas are related to the human being's tendency to organize and integrate the cognitions within a coherent and behaviorally efficient structure. Though the intuitive cognitions and models may help in the elaboration of internally coherent representations, these representations may be in contraposition with reality.

A human being has a tendency to integrate easily available information ignoring other information that would require a more sophisticated effort of investigation (Fischbein and Snarch, 1997). Fischbein and Grossman (1997) have suggested that intuitions are always based on structural schemas, that is to say, on organized systems of sequential interpretations and procedures which express a certain level of mental maturity and sufficient amount of experience.

## Intuitive mental models: the case of the unit impulse

When students have to face a notion of advanced mathematics that turns intuitively unacceptable for them, they produce, deliberately or unconsciously, more acceptable intuitive substitutes: the intuitive models, that may be understood, represented and manipulated like other concrete realities. (Fischbein, 1987). The use of a model involves thinking productively in terms of the model providing a simplified version of the reality. However, this can also present an inconvenience, if we consider the conclusions that the model used might suggest for the original. Usually we are not aware of many of the properties of the models used in a reasoning process, and, thus, they appear in an uncontrolled way.

Among intuitive mathematical models we will distinguish:

- The intra-mathematical analogies, in which an isomorphism is established between the original and the model despite their belonging to different conceptual systems.
- The paradigmatic models, in which the original consists of a class of entities, whereas the model is provided by an example or subclass of the considered category (Fischbein, 1987).

In one of our research works we have studied some difficulties and misconceptions of engineering students in the formalisation of the generalised function concept, which had been systematically observed in written and oral examinations and even in the application to the modelling of real situations. In particular, we studied the case of the unit impulse Delta function and its mathematical model within the distribution theory.

The study, carried out with 30 students in the third course of Mathematical Analysis included:

- a) The revision of textbooks of systems and signals usually consulted by students.
- b) A modelling activity, which was presented to the students with the aim of observing how they established the isomorphism between the mathematical theory and a physical model. It was related to the following situation:

*A ball is hit by a hammer on a plane horizontal surface, in such a way that it moves in a straight line and without friction.*

- c) 16 hours of mathematical instruction in relation to distribution theory.
- d) The analysis of the individual written respects about the modelling activity that students presented.
- e) A questionnaire about the impulse function and individual interviews with several students, which allowed us to study the students' beliefs, their ways of reasoning and the intuitive models evoked during the problem-solving.

From the results of these data we could conclude that:

- Students' showed high ability to apply theories and definitions when modelling a situation or solving a problem. However, this does not mean that the corresponding concepts have been internalised by them. This fact was observed during the interviews in which student had to justify their responses.
- The distribution model turned out to be intuitively unacceptable for the majority of the students considered in this study. Conceiving Delta as a function of a numerical variable—null everywhere except at zero, where it is infinite and with an integral different from zero—turned out to be intuitively more acceptable.
- Contradictions were ignored giving Delta a status of “Special” or “Different”. This last conception is encouraged by texts commonly used in Electronics, in an attempt to simplify. Yet, some of these texts usually offer an Appendix with a more rigorous presentation, which is usually ignored by students.
- The familiar model of a numerical function, assumed as paradigmatic, does not allow the acceptance of general ideas such as function spaces and functions defined on these spaces, even if an isomorphism has been previously established with signals taken as analogical models.

## Intuitive heuristics

Human beings have the tendency to organise and integrate cognitions—intuitive or logical and analytical—in a coherent and efficient structure. With development of age, experience and instruction, strong and stable beliefs are established. People rely on a number of heuristic principles, which reduce the complex tasks to simpler judgmental operations. In general, these heuristics are useful, but sometimes they lead to systematic errors.

We have studied some of these intuitive heuristics in relation to cognitive schemas and instruction: heuristics in probabilistic thinking (Tversky & Kahneman, 1983) and intuitive rules (Stavy & Tirosh, 2000).

## Heuristics in probabilistic thinking

In this study we have investigated many of the misconceptions mentioned by Tversky and Kahneman (1983) and by Fischbein and Grossman (1992), but considering the impact of instruction at university level and the evolution of the intuitive heuristics along the time. The most remarkable findings among our students were:

- Insensitivity to the sample size, revealed in questions like:

*The probability of obtaining 3 heads when tossing a coin 10 times is smaller, bigger or equal to the probability of obtaining 15 heads when tossing a coin 50 times.*



- The difficulty related to the effect of inversion in the time axis, consists in the insensitivity to the stochastic structure of a problem, under the influence of the causality principle and the irreversibility of time, which is deeply rooted in mental activity. It was revealed in questions like:

*There are two gold and two silver coins in a box. Peter takes out a coin and without looking at it. Then, he takes a second coin that is golden. The probability that the first coin was a gold one is smaller, bigger or equal to the probability it was a silver coin?*

In order to assess the evolution through instruction of these difficulties, we surveyed three groups of 30 students each one in different situations in relation to instruction: group 1: without formal instruction about probabilities, group 2: with formal instruction about probabilities but without final evaluation, group 3: a year later where they had passed the final exam of the subject.

A questionnaire with several problems aimed at detecting the above mentioned difficulties was delivered to the three groups, asking students to justify each one of their answers.

The results related to insensitivity to the sample size, showed a very low percentage of correct responses in the three groups (between 7% and 24%) and a strong influence of the belief that the size of the sample is not important. Proportionality was the underlying schema that was incorrectly used or overestimated to produce the incorrect responses. The highest performance was for group 2 and the lowest for group 3. This effect would mean that despite the instruction and passing of time, knowledge become less available and the person returns to strongly rooted intuitions.

In relation to the effect in the time axis, the results showed that only 48% of student could solve the problem correctly. The causality principle is the underlying schema: the idea that a fact cannot act retrospectively on its cause.

Students are strongly influenced by the order of events in the problem and this prevents them from realising the genuine stochastic structure. The inversion in the time axis, seems to contradict basic intuition and associated difficulties are persistent despite the presence of formal knowledge. However there is an improvement in students of group 3 (62% of correct responses), possibly because this group was formed by more mature students with a better training in mathematical modelling and formalization. This experience let them correctly handle the principle of causality.

## Intuitive rules

The theory of intuitive rules, developed by Stavy and Tirosh (2000), proposes that students' responses may be often determined by irrelevant external features of the tasks, rather than by underlying concepts.

Two of the studied intuitive rules are: "More A—More B" and "Same A—Same B". According to these rules, perceptual quantity A is taken as a criterion

for evaluating another quantity B, though the former is often not relevant to the required comparison or cannot, by itself, serve as a criterion.

The Rules More A—More B and Same A—Same B imply reasoning mechanisms, such as similarity, analogical reasoning, over-generalized logical schemas, but at the same time, they are considered intuitive ways of reasoning because they are accompanied by a strong feeling of self-evidence, confidence, perseverance, globality and coerciveness, which constitute, according to Fischbein, the distinctive characteristics of the intuitive reasoning. Several researchers have studied how these rules affect responses in mathematics and science students, but most of those studies were developed with rather young students.

In order to study the influence of intuitive rules on engineering students dealing with advanced mathematics, the following problem was presented to 30 students. Such a problem could be modeled and solved using formal knowledge about differential equations conveyed during the usual lessons.

*A buried tank contains an unknown volume of salted water. It is suspected that the tank is leaking, spoiling the environment. Though it is not possible to measure the volume or the quantity of salt directly, it is possible to inject and extract water at the same rate of 1 gal/min.*

*The first extraction reveals that the initial concentration is 0.2 lb/gal.*

*The injected solution has a concentration of 0.05 lb/gal.*

*It is assumed that the solutions mix instantaneously in the tank.*

*Several data of outcome were given, and students were requested to model the problem mathematically, to solve the differential equation and find the function “quantity of salt”, considering the possibility of a leakage. They should also model and solve a similar situation where there is no leakage.*

Students presented a written report with the solution for both situations.

After a week, they were interrogated again about the problem, but this time, they were asked to make a qualitative analysis regarding the evolution of volume, concentration, and quantity of salt in the tank (that is to say, if these variables increase, decrease or keep constant) in both cases: leakage (L) and no leakage (NL).

8 students were also interviewed and interrogated about the same issues.

Results showed that although 94% of the students could solve the problem correctly in both situations (L and NL), when they were asked about the evolution of the processes, many of them were influenced by the intuitive rules focusing the attention on one of the variables only.

For instance, in the evolution of concentration, despite the fact that 96% of the students considered that the concentration diminishes in the L case, only 20% of them considered both effects in the analysis: dilution and volume decrease. The rest of them focused on only one of these variables.

In the NL case, 64% of the students considered that the concentration diminishes. That is to say, there were 32% of them who thought that concentration diminishes in the L case and does not diminish in the NL case. Among them, a half consider that the concentration keeps constant because “*the volume keeps constant*”. These answers are clearly in line with the intuitive rule Same (volume)—Same (concentration). The other half believed that the concentration increased because “*we are adding salt*”, and thus, they forgot about the dilution effect. In this case, they reacted according to the rule More (quantity of salt)—More (concentration).

In the analysis of the evolution of the quantity of salt, students had even more difficulties. A higher percentage of the students did not answer or did not account for their responses.

Again, in this situation, there were 16% of the students who associated the evolution of this variable with the volume (Same (volume)—Same (quantity of salt)). They considered that the quantity of salt keeps constant in the NL case because “*the volume doesn't change*”. There were also 24% who considered that the quantity of salt increases because “*we are adding salt*”, neglecting other effects.

## Final remarks

Intuitive reactions are strongly influenced by the experience and psychological profile of each individual. However, certain patterns related to intuitive models and intuitive heuristics can be identified when students deal with certain concepts and problem-solving tasks in mathematics. These intuitive ways of reasoning may create difficulties in the acquisition and application of mathematical knowledge.

The study of the evolution of intuitive schemas in relation to complex pieces of mathematical knowledge becomes paramount at university level.

Logical schema and bodies of mathematical knowledge are expected to develop and strengthen with age and/or instruction. Consequently, intuitive reasoning may lose its power in favor of other competing knowledge. However, in this study we could observe that some intuitive heuristics are resistant to age and experience evolution.

In the students' responses to certain mathematical tasks, we could observe that, in each intuitive response there seems to be an underlying logical schema, or a mental model intuitively accepted by the student, which interacts with the specific restrictions of the presented problem.

In the problem of the Delta function, a more acceptable intuitive model was developed by the students, who ignored the contradictions and focused their reasoning on some specific properties of that model.

In probabilistic thinking, the logical schemas of proportionality and causality influenced the students reasoning strongly.

In the problem modeled by a differential equation, the responses were influenced by intuitive rules that might have been the result of a more general tendency to extrapolate given information to new situations.

In all these situations, relevant information and pieces of mathematical knowledge were neglected and students focused on some variables only.

In order to design didactical interventions when dealing with concepts of certain complexity, it seems highly recommendable that teachers which be aware of the students intuitive models and intuitive heuristics.

Modelling activities in real situations seem to be suitable instances for helping students become aware of their own intuitive models. These activities could help students to develop meta-cognitive attitudes.

Problems like the ones posed in these studies, which are known to elicit intuitive responses, should be presented to students in order to produce a cognitive conflict between their beliefs and the associated formal knowledge. The solution to this conflict, the analysis of the problem structure and the source of possible errors produced by intuition could help them overcome these difficulties and possibly generate more adequate intuitions.

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# Operations Research for Engineers: Two-cycle Approach

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## 5 Formulation of the problem

Mathematical education of engineers in the two-cycle system brings several methodological problems. One of them is the question of choosing the appropriate extent and depth of the teaching material, which apparently should be different for bachelors and for engineers, according to different educational goals in both categories.

One part of the students will finish their study after the first cycle. For them, the mathematical education should provide a solid background for practical applications. Another part of the students will continue the study in the second cycle. These students must be prepared for applications, as well, but they will also need a basic theoretical knowledge in their future work, in order to understand the available literature and to use it creatively. There is also the third category of students: the bachelors coming from other universities, wishing to continue their study at master level and become engineers.

The question stands as follows: can mathematical subjects in the first cycle be taught in the same way for the first and the second group? and: can the mathematical subjects in the second cycle be taught in the same way for both the second and the third group?

The basic question in other words: is it possible to organize the mathematical education in the first cycle in such a manner which would be appropriate at the same time both for the future bachelors and for the future engineers? and analogously for the second cycle.

In which mathematical subjects is this possible? How to do it in an optimal way?

## 6 Goals of the mathematical education for engineers

This presentation describes one possible solution to the above problem of the two-cycle approach, demonstrated on the subject “Methods of Operations Research” at the bachelor and engineer study levels, in notation: MOR 1 and MOR 2, respectively. The ideas are based on the experience at the Faculty of Informatics and Management, University of Hradec Králové, in the Czech Republic.

Design of the lectures, exercises and exams on both levels of the subject are based on the goals of the mathematical education in general. According to the author’s opinion presented on former SEFI seminars, a bachelor or an engineer with the proper mathematical education should:

- be convinced in the usefulness of mathematics for his/her work
- understand the corresponding mathematical notions in some necessary extent
- be able to use the mathematical methods adequately
- be aware of the limited applicability of the methods
- know which of the mathematical methods is the most appropriate in a given situation
- be able to recognize situations, when an approximation is better than the exact computation
- perform the mathematical methods accurately
- be able to discover and correct the mistakes in the computations

We have tried to organize the teaching of the both subjects MOV 1 and MOV 2 in such a way that the differences in the mathematical educations between the first cycle (bachelors) and the second cycle (engineers) will be taken into account.

## 7 Methods of Operations Research 1

Every lecture on MOR 1 begins with a motivation example that indicates which type of problem will be considered. The simple example shows the tension between the desired result and given limitations in the presented situation. Then a solution of a special case is found and a discussion follows, on using the similar solution idea in analogous situations. The discussion leads to a general method, applicable to a defined class of problem.

For a more exact formulation of the method found in the first part of the lecture, and for the description of the problems, which can be solved by that method, necessary notions must be introduced, and their properties must be studied. Mathematical models of the problem and the mathematical notation are useful and efficient tools for this purpose. Finally, various examples are computed, showing details and

limitation of the explained method, as well as the conditions, which are necessary for proper and reliable application.

## 8 Exercise to MOR 1

The exercises take place in computer laboratories. Every student is connected to the university web, where the texts of the examples can be found. The texts and the necessary data are written in Excel tables, into which the student writes his/her own solution. In this way, every student is systematically creating a personal database of solved examples, which he/she can later use in preparation for the exam at the end of the semester. As a feedback, the students have at their disposal a set of solutions prepared by the teacher.

A weekly set of examples for the exercise is accompanied by another set of examples for the individual work at home or in computer laboratories. Another folder at the university web contains the teaching texts, which were projected during the previous lectures. As a consequence of this arrangement, every student advances in his/her own rate of study.

The main purpose of this cycle is to give a clear idea of the problems and of how the solution methods work. For this reason the given problems are typical ones, they are not too large and the Excel Solver is not used in this cycle.

## 9 Methods of Operations Research 2

The principal arrangement of the lectures and exercises in the second cycle is similar to that in MOR 1. The basic ideas from the first cycle are assumed to be preliminary known (this concerns mainly the third category of students: the outside incomers). To provide the students with a deeper knowledge of the subject, more advanced introductory examples are used as motivation. More complex problems are solved, showing a variety of possible subtle complications and variations. The considered methods are more complicated and more powerful than they were in the first cycle. The examples contain larger amount of data, and therefore the Excel Solver is used at the lecture and at the exercises, as well.

The main purpose of MOR 2 is teaching the students to use the appropriate methods and to use them properly. The logical analysis of a given problem and the correct interpretation of the results provided by the computer solver are of central importance. The advanced problems are not only larger and more richly structured than in MOR 1 but the dependence on changing parameters and/or conditions is also considered.

## 10 Exams on MOR in computer laboratory

Performing exams in computing laboratory has several advantages. In our MOR teaching system the conditions on the exam are equivalent to those on the exercise. Computers help students to concentrate their attention mainly to methods, and

less to computations with numbers. Computers save valuable time which otherwise would be wasted by performing automatic operations on input data. The output data are computed quickly and precisely. Use of computers makes it possible to work with larger and more complex problems in reasonable time.

On the other hand, the computers connected to the university web and to internet provide large possibility of worldwide connections, and in the consequence, they enlarge the risk of cheating attempts. One way out of this unpleasant situation would be taking repressive measures: disconnecting computers off the web during the exam, supervising strictly the students, creating the atmosphere of fear so that students would not dare to cheat. We do not support this negative attitude to students.

Our approach to eliminate the undesirable behaviour of the students is based on several simple exam rules, under which the most efficient way of doing the exam is the individual independent work. Our exam rules make any attempts of misusing the web capabilities less efficient and relatively too much time-consuming.

The basic exam rule is that the exam tasks are distributed to students in a printed form, and the answers (results of the computation) must be written on the indicated positions on the exam paper. Further, it is forbidden to transform the complete text of the tasks to an electronic form and to give a task, or a part of it, to other persons.

The teacher can easily maintain a relatively large non-public database of exam tasks in the electronic form, with possible creation of many variants, and quick computation of the correct answers by a computer solver. Thus, practically every student in the exam room gets a different sheet with his/her individual tasks.

The positive approach to students is expressed by the rule allowing them to use their personal databases of solved examples, and even more, to use every teaching material placed on the web. As the time for the exam is not very large, and the student is the only person who knows the tasks written on his/her exam sheet, the study materials can efficiently be used only if the student really understands them.

## 11 Computer-aided education at FIM UHK

Several remarks concerning computers in education will be presented in this final section. Involving computers in teaching the Methods of Operations Research in two cycles, as described above, is nothing else than just using the relatively good computer equipment at the Faculty of Informatics and Management of the University of Hradec Králové. We may characterize it as “computer-aided education”. For a teacher it means using computers for more efficient preparation of the standard teaching materials and for some extension of the standard teaching methods. To a student, the above use of computers means the possibility of choosing an individual style of learning and a time independent access to the subject.

In the last years, there is a high effort to prepare e-courses at FIM UHK. At present time, 60 e-courses on various subjects have been allowed for use in the regular teaching process, and a large number of further e-courses, among them MOV 1 and MOV 2 are in preparation. Other 15 e-courses are used in the life-long learning



system provided by the faculty. As a rule, the e-courses at FIM UHK are prepared in the e-learning system WebCT. The advantages of e-courses for the students are: the feedback, dialogues with teacher, self-evaluation tests and further facilities. The experience with e-teaching and e-learning are systematically evaluated, so that the positive aspects, as well as the negative ones, can be taken into account in creating further e-courses in future.

# The Scientific Interests of a Lecturer Influence the Process of Mathematical Learning

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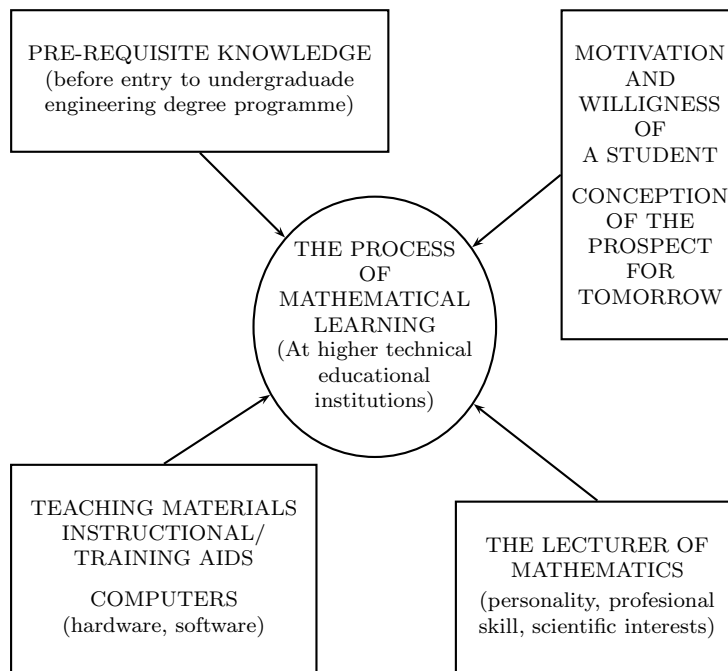
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Considerable effort is made and much time is spent trying to contribute to the development and improvement of basic mathematical learning at higher technical educational institutions in Europe. The regular seminars of the SEFI Mathematics Working Group (MWG) provide a basis for serious and valuable discussions concerning the applicability of computers and software packages to mathematics teaching. These have led to the appearance of the original and the revised versions of the SEFI Core Curriculum in Mathematics.

It goes without saying that all the efforts and further discussions on the matter would be doomed to failure unless both parts of the “student-lecturer” link are adequately considered. Evidently, the former link (the student) should be prepared and well disposed towards mathematical studies. The willingness of a student is a necessary pre-condition for the successful mastery of the mathematical truths. But, it is not sufficient. We are well aware that engineering students cannot perceive mathematics in the same way as professional mathematicians usually do.

So, we are about to draw your attention to the lecturer of mathematics (the second link; see Fig. 1). Great demands are made of him, namely:

- teaching ability (psychological preparedness);
- pedagogical talent;
- excellent understanding of the subject;
- desire for knowledge, desire for being perfect not only in the teaching of mathematics, but also in one or another scientific research area (relevant to pure or applied mathematics).



**Fig. 1.** Many factors influence the process of mathematical learning

The latter circumstance (demand), to our mind, is of utmost importance (Figure 2). Firstly, scientific research activities grant the lecturer of mathematics a right to bear the status (halo) of a scientist. Secondly, the lecturer of mathematics, who is engaged in progressive scientific work, will, undoubtedly, be better prepared to answer questions of the audience (curious engineering students). The lecturer-scientist will always be ready to give serious explanations, based on his own experience, to the most traditional freshmen’ remark—“What is the good of mathematics at all? Why is a knowledge of mathematics (mathematical techniques) essential for their future practical work?”.

Evidently, the path of self-perfection can be neither an end in itself nor isolated. Labour-consuming activity, personal contacts, interchange of opinions and views are the necessary prerequisites. In their turn, scientific interests, scientific research work as well as the scientific achievements of the lecturer stimulate cooperation and partnership among mathematicians-scientists in pursuing the implementation of various tasks—international research projects, international programmes (including students/staff mobility), etc.

To try to clarify the real influence of the scientific research activity of the lecturer on the quality of delivery of mathematical courses, we have distributed a questionnaire to a number of undergraduate and graduate students of the Faculty of Fundamental Sciences (Kaunas University of Technology). The respondents (55, in total) were asked to estimate (on a five point scale) the topicality of some mathemat-

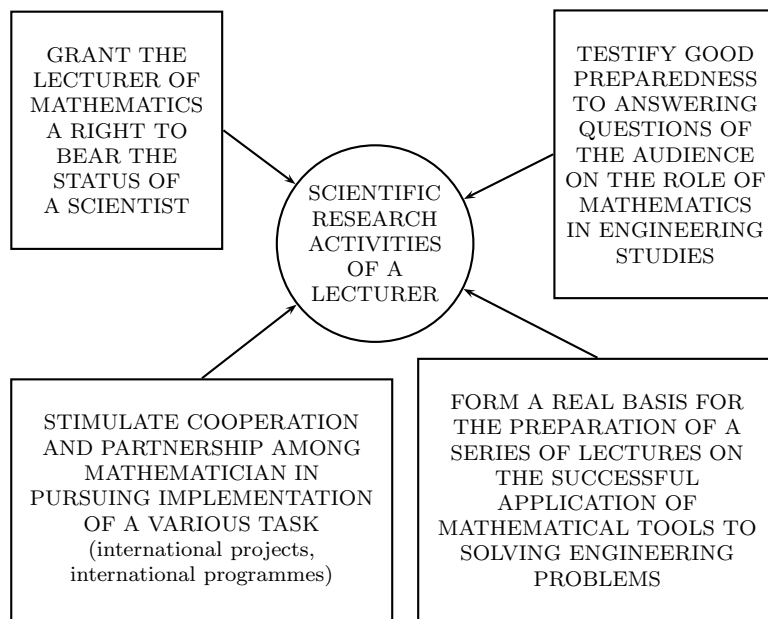


Fig. 2. The scientific research activities of the lecturer of mathematics are of great use

ical courses (20 study modules) delivered by lecturers of the Department of Applied Mathematics, to estimate (variable  $X_1$ ) the presentation of the course material in the context of up-to-date research achievements (within the country, world-wide; variable  $X_2$ ), to estimate the efforts of lecturers to illustrate the application of a particular mathematical tool (technique or approach, associated with the course material) to solving actual engineering problems (variable  $X_3$ ) and, finally, to estimate the attempts of lecturers to relate the course material with their own scientific research attainments (variable  $X_4$ ).

Also, the board of experts from the faculty of Fundamental Sciences (Kaunas University of Technology) has prepared qualifying appraisals (on a five point scale; variable  $Y$ ) for lecturers included in the evaluation. The qualifying (rating) criterion for each lecturer was determined by the number of serious scientific publications (International Journals, Conference Proceedings), and the level of supervising activity.

Statistical analysis results revealed a positive correlation between moderate scientific research activity of the lecturer of mathematics and the quality of presentation (from the student's standpoint) of basic and specialized mathematical courses. To be more precise, the Spearman's correlation coefficients for ranked data took the following values:  $\rho(X_1, Y) = 0.5$ ,  $\rho(X_2, Y) = 0.57$ ,  $\rho(X_3, Y) = 0.64$ ,  $\rho(X_4, Y) = 0.75$  and are significant at the 0.05 level of significance.

Also, it was found that the scientific interests of the lecturer facilitate better conveying of the contents of mathematics study modules to the audience, and promote better understanding of mathematics on the whole.

We propose here one more idea to change things for the better. It is a questionnaire for the SEFI–MWG correspondents. Everybody, who is engaged in the delivery of mathematical courses at higher technical educational institutions, is kindly asked to visit our web site `fmf.ktu.lt\maths`, and to fill in the questionnaire there, i.e. to give short answers to straightforward questions (scientific research area, research object, specialized mathematical courses delivered by a respondent, etc.).

We think that the data collected will form a real basis for the preparation of a series of lectures (within the SEFI framework) on the successful application of mathematical tools to solving a range of engineering problems. Later on, such material (lectures) could be disseminated, using the Distance Education Learning Networks, to those students who perceive that problem solving is a good test of understanding mathematics and who intend to find the right balance between practical application of mathematics and in-depth understanding.

At the same time, it would be a real support for many mathematics departments not only in realizing their own curricula, but also in opening cooperation with other academic institutions.

# The Bologna Process in Germany and at the TFH Berlin

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*The Bologna process in Germany has increased speed since the Berlin follower conference in 2003 as on the next conference in 2005 in Bergen a report of the achieved goals has to be given.*

*Now all universities and universities of applied sciences seriously start to switch into practice all their traditional degree programmes “Diplom” respectively “Diplom (FH)” into the Bachelor-Master-scheme. Especially the professional qualification of Bachelor’s degree programmes is discussed intensively and differently within the given regulatory framework. This includes the mathematical education of engineers. As a typical example of this process the situation at TFH Berlin will be described. Everything is turned upside down. A lot of unsolved issues remain.*

*On the other hand the quality assurance system has been established with the National Accreditation Council (Akkreditierungsrat), and the accreditation agencies are successfully working. As an elected member of the accreditation commission of ASIIN (Accreditation Agency for Study Programs in Engineering, Informatics, Natural Sciences and Mathematics) aspects of the quality assurance will be presented.*

## **BA/MA—extreme acceleration (1994–2005)**

“Bologna” marks not only the creation of the university as the oldest type of higher education in 1119 but above all the agreement made in 1999—legally without obligation—by at that time 29 and now 40 Ministers responsible for Higher Education to realise a “European Higher Education Area” until 2010. The main aims of this “Bologna process” are the improvement of the mobility of students and teaching staff as well as the strengthening of the competition of the European universities in a global education market. In fact this process started already in 1994 where the General Agreement on Trade in Services (GATS) has been signed on a conference of the World Trade Organisation (WTO) [3]. Their members agreed to reduce

the obstacles in the trades of services including educational services. This agreement opened the international competition in the area of university and continuing education. Next, in 1998 France, Germany, Italy and United Kingdom agreed to establish a common system of higher education. In the same year the framework law of higher education for the university system has been passed in Germany that allowed introducing Bachelor's and Master's programmes.

In 1999 in Bologna the minister of 29 countries agreed to create a comparable university system maintaining the cultural richness and linguistic diversity, based on its heritage of diversified traditions. Seven main aims have been announced in Bologna to be realized: (1) a system of transparent comparable degrees by a Diploma Supplement, (2) a two tier system of study degrees, (3) a credit point system (ECTS) and modules, (4) student mobility, (5) a European cooperation in quality assurance, (6) a European dimension of university education, and (7) a life long learning. In 2001 the first follow-up conference took place in Prague. The follow-up conference in Berlin has been joined by 33 European countries and three new aims have been added: (8) the participation of students in the Bologna process, (9) the enhancement of the attractiveness and competitiveness of the European university system, and (10) the inclusion of PhD-studies. The Bologna process was also expanded to 40 European countries. Furthermore the ministers agreed in Berlin to check at the next conference in 2005 in Bergen whether the following three goals have been achieved: (1) the Diploma Supplement, (2) the start of BA/MA programmes and (5) the accreditation system.

A year ago only a few Bachelor's and Master's programmes existed at German universities. Their existence has been mainly due to isolated initiatives of engaged members of the staff.

A general discussion did not take place in the academic committees or in the public. Neither practical impacts nor the teaching methodology has been discussed. These early new programmes are mainly offered in addition to the existing degrees and the main focus of interest was on postgraduate and continuing Master's programmes. The expectant and reluctant interest in the new BA/MA programmes was associated with the hope that the existing degrees could be kept and offered in parallel to the new ones. Since the Berlin follow-up conference of the Bologna process in September 2003 this hope has been destroyed by the German Secretary of State for Higher Education. The feeling "who is coming late is punished by life" has lead to an excessive desire for action of the proper authorities in Germany. The existing BA/MA initiatives now lead to a chain reaction. The switch into practice is not centralized; the federal states (Bundesländer) and the different subjects ensure diversity. The proper authorities believe in the regulatory mechanisms of the educational market.

At German universities, including the universities of applied sciences, there exist 5011 traditional degrees leading to a Diplom, Diplom (FH) or Magister (in arts) and 2851 degrees for the education of teachers [2]. On the other hand there exist 2445 BA/MA programmes [2], out of which 493 BA/MA programmes have been accredited by the National Accreditation Council (Akkreditierungsrat), and for further 645 BA/MA programmes the process of accreditation is running [1]. However, only

about 6% of all students study in these new BA/MA programmes mainly because smaller subject had switched to them.

## BA/MA regulatory framework in Germany

The latest issue of the conference of ministers for the arts and culture (Kultusministerkonferenz) has fixed the following framework in 2003: Bachelor's programmes can last 6–8 semesters leading to the degree Bachelor of Arts, Science or Engineering (B.A., B.Sc., B.Eng.). Students shall gain a scientific basis and a competence in methods of their subject and thus shall be qualified for an academic profession. The Bachelor's degree shall be the main university degree, i.e. 50%–80% of the Bachelor's graduates will directly take up a job in industry. Master's programmes can last 2–4 semesters leading to the degree Master of Arts, Science or Engineering (M.A., M.Sc., M.Eng.). The admission requirement for a Master's programme is a qualified Bachelor's degree. Master's programmes can be research or application oriented. The Master's degree qualifies in general for the admission to a PhD study. The workload of the students is measured in credit points according to the European Credit Point System (ECTS). One semester consists of 30 credit points being equivalent to about 900 hours of student work (participation in lectures, laboratory experiments, seminars and home work or self-study), i.e. 1 credit point means 30 hours workload for the student. A Bachelor's programme of 6 or 7 semesters consist of 180 respectively 210 credit points. Consecutive Bachelor's and Master's programmes last together 10 semesters and consist of 300 credit points. The profile of the programme has to be indicated in the Diploma Supplement. There are three different types of Master's programmes: consecutive, non consecutive and additional Master's programmes. A consecutive Bachelor's and Master's programme has a duration of 10 semesters with 300 credit points. However, the portion of each part can be different: 6 + 4, or 7 + 3 or even 8 + 2 semesters, depending on the profile of the programme. By changing the university it is possible that a student studies a 7 semester Bachelor's and then a 4 semester Master's programme or a 6 semester Bachelor's and then a 3 semester Master's programme. A non consecutive Master's programme doesn't have a specific Bachelor's programme of the same subject but allows typically different Bachelor's degrees as entry requirement. The additional Master's programme aims more to the idea of lifelong learning. It requires an industrial experience of at least one year. Furthermore the duration of the final project (thesis) has been fixed: 6–12 credit points, i.e. 20%–40% of the workload for a whole semester, have been assigned to the Bachelor's project. 15–30 credit points, i.e. 50%–100% of the workload for a whole semester, have been assigned to the Master's project.

The quality assurance of the BA/MA programmes is guaranteed by the accreditation. The National Accreditation Council (Akkreditierungsrat) is responsible [1] which has been set up in accordance with the resolution of the Standing Conference of the Ministers of Education and Cultural Affairs of the federal states (Länder) in the Federal Republic of Germany (Kultusministerkonferenz—KMK). As an independent institution the Akkreditierungsrat is made up of 17 members, who are representatives of the Länder, higher education institutions, students and profes-



sional practice. There are six accreditation agencies that carry out the work. These agencies set up minimum standards for the programmes. A new aspect is that they require a job statistic of the graduates. The accreditation is limited to 5–8 years. Of course the universities have to pay to get the Accreditation Council's Quality Certificate. At the moment the amount is 12.500. There is no formal or legal difference in accredited BA/MA programmes for the two different types of universities in Germany: Fachhochschule (University of Applied Sciences) and Universität (University). Concerning the sections of the civil service a relation between the new and old degrees has been fixed: A Bachelor's degree entitles as the Diplom (FH) degree for higher while a Master's degree entitles as the Diplom degree for senior sections of the civil service. After 2010 no student will start his/her study in the the old Diplom programme.

## Existing dual university system in Germany

Germany has a dual university system [2], 122 Fachhochschulen (University of Applied Sciences) and 99 Universitäten (University/Technical University). The 14 Technical Universities refer to a special tradition. Germany has nearly 2 million students. 25% of them study at a Fachhochschule. The main differences between Fachhochschulen und Universitäten summarize up to:

After 12 years of school (Fachhochschulreife) one can study at a Fachhochschule (FH). All programmes are designed for 8 semesters and this study time is also realistic. All the programmes are application oriented. Very good graduates from a Fachhochschule can take up a PhD study at a Universität. At a Fachhochschule students study in small classrooms of 40 from the very beginning of their study. The assessment method is cumulative, i.e. every semester in every subject. Lectures and also exercises are both done by professors, who have a workload of 18 hours per week. All professors have industrial experience of at least 5 years that is acquired before they become professor.

The admission to a Universität requires 13 years of school (Abitur). The programmes are designed between 8 and 10 semesters depending on the subject but the average study time is much higher. The programmes are research oriented. The Universität has the right to deliver the PhD grade. There are huge lecture groups in the basic subjects. Lectures are given by professors and exercises are given by post-graduates and by students. In general at a Universität you don't have cumulative assessments. Professors at a Universität have a distinguished research qualification. Their workload is 8 hours per week.

## Chances and risks of the new BA/MA system

The chances and risks of the new BA/MA system are different for the two types of universities. For a Fachhochschule it is difficult to maintain its special profile within a Bachelor's programme of 6 semesters. On the other hand a 7 semester Bachelor's programme might have a public relations problem. A positive aspect is that Master's

programmes are possible and that a Master (FH) doesn't exist. However the switch from Diplom programmes of 8 semesters to consecutive BA/MA programmes of 10 semesters requires additional money. So there is a distortion in the competition between Fachhochschulen and Universitäten. The idea of a Bachelor's programme transfers the special profile of Fachhochschulen partly to Universitäten. However the workload of professors is quite different at these two types of institutions. For the Universitäten Bachelor's programmes are possible and they open a chance for weaker students instead of dropping out university. However, how will universities design a Bachelor's programme of 6 semesters that qualifies for a job? The Universitäten prefer research oriented Master's programmes of 4 semesters. Without requiring extra costs it is possible to switch from a 10 semester Diplom programme to a consecutive 10 semester BA/MA programme. For the Universitäten a complete redesign of the existing programmes is necessary because a Bachelor's degree shall qualify for a job and a change in the assessment system has to take place.

One common problem exists for the basic subjects like mathematics in engineering programmes. They will be reduced in the Bachelor's programmes. But will there be advanced mathematics in the Master's programmes? Universities with many students have the best chances with the new BA/MA system and also programmes with many students like computer science are in a good position. A great danger consists in the probable misuse by politicians because they might further shorten the money for universities. 50%–80% of students leave universities with a Bachelor's degree after 6 or 7 semesters. The switch from contact hours to credit points (workload for students) could be misused to calculate the staff capacity at a new basis. There are two currencies: professors are paid in contact hours (18 or 8 per week) and students are paid in credit points (900 hours personal workload per semester). Will there be in the future mathematics modules of 5 credit points but with only 1 contact hour per week? A didactic discussion is completely missing in the Bologna process where public discussion mainly concentrates on the output, i.e. learning outcome, but the politicians mean the reduction of government funding.

## Switch into practice at TFH Berlin

The Technische Fachhochschule (TFH) Berlin has decided to keep the special profile of a Fachhochschule. This is in awareness of the fact that the 14 Technical Universities in Germany (Aachen, Berlin, Braunschweig, Clausthal, Chemnitz, Cottbus, Darmstadt, Dresden, Freiberg, Hamburg-Harburg, Ilmenau, Karlsruhe, Kaiserslautern and München) seem to have the opinion that the qualification of an engineering profession can only be achieved after 10 semesters with the Master's degree and that a 6 semester Bachelor's degree would be more or less like a technician. The Fachhochschulen know that the 8 semester Diplom (FH) qualifies for a job, especially for an engineering profession. This is achieved by an internship of one semester, the final Diplom thesis of one semester and the small learning groups of 40 Students from the very beginning of their studies.

So the TFH Berlin prefers the 7 semester Bachelor's programme. However there are exceptions in computer science with a 6 semester Bachelor's programme. There

will be an internship of 15–25 credits and a final Bachelor's project of 12 credits. The time of switch into practice will be individual but before 2010. TFH Berlin will also offer some Master's programmes. This is possible without any additional staff for (6 + 4)-programmes like computer science where 80 students are studying in the Bachelor's and 40 are continuing in the Master's programme. The other programmes will have to create staff capacity by reducing the contact hours but keeping the work load for the students. The calculation of staff capacity in the existing Diplom (FH) programmes is done on the basis that during the 6 theoretical semesters (except internship and thesis semester) there are about 27 contact hours per week for a group of 40 students. Instead of reducing the contact hours in the Bachelor's programmes it is possible in Berlin to get in some cases more government funding because the federal state Berlin shifts some capacity from Universitäten to Fachhochschulen. A third possibility would be the introduction of study fees which is now only possible for non consecutive continuing Master's programmes.

At TFH Berlin there exists the tendency to reduce the quality of Mathematics modules by introducing of BA/MA programmes. One keeps the mathematical contents and the workload for the students but offers less contact hours of students with professors. This will probably be done although the students come with less mathematical knowledge from school. The consequence is that the students have to learn more at home from books and that the professors have to supervise more students. The engineering colleagues also discuss the fact of creating combined modules with mathematics and physics or mathematics and technical mechanics. This idea inherits the problem of transferring credits to other universities where this special type of combined module doesn't exist. The didactic aspect is that mathematics would then be taught by engineering colleagues only in the special context of their subject. A third point is that there might be no mathematics at all in engineering Master's programmes.

## **BA/MA—a potential for economizing**

At the first glance the Bologna process means a switch from the Diplom degree to a two tier BA/MA degree. In reality it is means of reducing the government funding. This is absolute because there will be a shorter study period for the majority of students. Secondly the university system will be transformed in direction of a private company. At the moment a new salary system for professors is being introduced in Germany. The academic self-government is going to be changed. The rivalry between Fachhochschulen and Universitäten will remain but the competition is distorted. In addition industry will pay less starting salaries for Bachelor's graduates but will probably require the same quality. This process goes in parallel with the General Agreement on Trade in Services (GATS) that has been signed in 1994 and that opens an international competition in the area of university and continuing education.

*“Das Alte stürzt, es ändert sich die Zeit,  
Und neues Leben blüht aus den Ruinen”*

Friedrich Schiller: Wilhelm Tell (1804)

*“The tradition falls, it changes the time,  
And new life flowers out of the ruins”*



Modern performance in  
Altdorf/Switzerland 1998

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# Teaching of Mathematics and Mathematical Modelling Using Computer Applications

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## Abstract

The Panevezys Institute of Kaunas University of Technology hosts students of mechanical technology, civil engineering, and electrical engineering. The level of proficiency required in different mathematical topics differs slightly for different specialisms. But engineers of all specialisms need mathematical knowledge for the solution of practical problems and for research in the fields of mechanics, control, physics etc.

Rapidly developing computer technologies enable us to use mathematical modelling widely in a range of technical areas. So the students should be familiar with the development of a model and be able to choose a suitable algorithm and software, and be able to write simple add-ons to that software for solving the model. This paper discusses the implementation of these aims.

## 1 Introduction

A different level of mathematical knowledge is necessary for students of different specialisms. We are analyzing here general aspects of teaching mathematics to students from different technical specialisms (engineers to be). Naturally, mathematical knowledge is needed for an engineer for solving particular problems as well as theoretical research closely connected with practice: physics, mechanics, engineering, control, etc.

Owing to the rapidly expanding applications of mathematical models and computer technologies in all technical sciences, there is a necessity for a specific character of teaching mathematics. Requirements of the mathematical knowledge to be met by a modern engineer are formulated in the SEFI Mathematics Working Group Core Curriculum [1].

Having a specific mathematical education, an engineer must know how to

- state mathematical problems,

- construct mathematical models,
- select a proper mathematical method and algorithm for solving a particular problem,
- implement the selected algorithm by computer (by directly programming or applying a software package of general or special purpose),
- apply knowledge of pure mathematics (qualitative mathematical investigations),
- define the adequacy of the mathematical model selected,
- give efficient recommendations based on the mathematical analysis.

Bachelors of civil engineering, mechanical engineering, and electrical engineering, masters of mechanical engineering as well as bachelors and masters of business and management specialisms are prepared at the Panevezys Institute of Kaunas University of Technology. Daytime department engineers study mathematics for 3 terms, and those of the evening department for 2 terms, starting from the first term of the first year. Informatics is studied in the first and second terms. In the second term of informatics the application of computer algebra packages is covered with practical exercises in MathCad.

The instructor usually has three options for the lecture:

- a) to consider only a mathematical topic and to solve problems “by paper and pen”;
- b) to consider only a mathematical topic and to solve some of the problems “by computer” prior to having learned how to solve them by “paper and pen”;
- c) while considering a mathematical topic, to formulate a real life problem, to develop its mathematical model and to solve it by the first or second method.

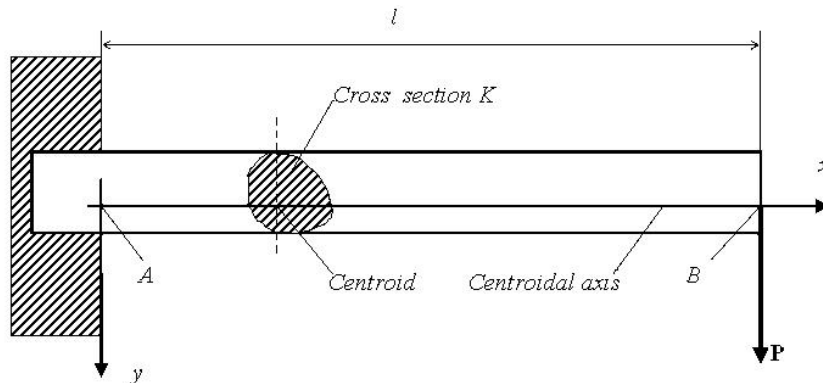
The third way is most useful for an engineer, however, it is time consuming and requires knowledge of mathematical methods. Also, development and research of mathematical models do not serve as substitutes for normal lectures on mathematics.

## 2 A Sample Problem

We present here a typical situation that completely reflects the above-mentioned requirements. The model is based on a differential equation (such equations frequently serve as a mathematical model of real processes).

### 2.1 Real life problem

Consider a horizontal beam acted upon by vertical loads. Assume that the forces due to these loads lie in a vertical plane containing the neutral (centroidal) axis of the beam and that they are such that no part of the beam is stressed beyond its



**Fig. 1.** A cantilever uniform beam

elastic limit. These stresses cause the beam to bend, as indicated in Fig. 1, and the curve of its neutral (centroidal) axis is called the elastic curve of the stressed beam. If the beam is made of uniform material satisfying Hooke's law, and satisfies certain other conditions related to the shape and properties of materials, it can be shown that its elastic curve approximately satisfies the differential equation

$$EJ \frac{d^2 y}{dx^2} = M. \quad (1)$$

Here the axis  $x$  is horizontal along the beam, the axis  $y$  is vertical,  $E$  is the modulus of elasticity of the material of the beam,  $J$  is the moment of inertia of the beam perpendicular to its axis with respect to a horizontal line in the cross section  $K$  passing through its centroid, and  $M$  is the bending moment in the cross-section. Since the material of the beam is uniform,  $E$  is constant, and if the beam has a uniform cross-section,  $J$  is constant. Fig. 1 represents a cantilever uniform beam  $l = 6\text{ m}$  long, fixed at one end (point  $A$ ) and carries a concentrated load  $P = 20000\text{ N}$  at point  $B$ . Let  $J = 3 \cdot 10^{-4}\text{ m}^4$  and  $E = 2.1 \cdot 10^{11}\text{ Pa}$ . In addition, some measurements have been made at the points,  $\bar{x}_i = i$ ,  $i = 3, 4, 5\text{ m}$  whose data are

$$\bar{y}(3) = -0.00783, \quad \bar{y}(4) = -0.00991, \quad \bar{y}(5) = -0.01920.$$

We have to find the deflection of the beam at each point of the interval and to estimate the adequacy of the model.

## 2.2 Construction of the model

The bending moment  $M$  at any cross-section may be found by taking the algebraic sum of the moments of external forces on a part of the beam on one side of the cross-section about the horizontal line in this cross-section. In search of  $M$ , we

consider upward forces giving positive moments and downward forces giving negative moments (this is the sign convention). In this case (Fig. 1)

$$M(x) = P(l - x) \quad (2)$$

and from (1) we get a differential equation for deflection

$$EJ \frac{d^2 y}{dx^2} = P(l - x). \quad (3)$$

Since the beam is fixed at one end  $A$ , the elastic curve is horizontal at this end. Hence, taking the origin on the left end  $A$ , we have initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 0. \quad (4)$$

### 2.3 Exact solution of the model

Integrating (3) and substituting the conditions (4), we obtain the exact solution

$$y(x) = \frac{P}{2EJ} \left( lx^2 - \frac{x^3}{3} \right). \quad (5)$$

### 2.4 Numerical solution of the model

By a numerical solution of a differential equation in  $x$  and  $y$ , we mean a table of the values of  $y$  opposite to the corresponding values of  $x$  for a particular solution of the equation. Though equation (3) with its initial conditions (4) can be solved analytically (such a choice is for methodical considerations), we suppose that there is no analytic solution, because that is a typical situation in real problems. This remark also applies to equation (1) under consideration, since, if deflections are more significant, the neutral line is described by the nonlinear equation

$$EJ \frac{d^2 y}{dx^2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{-\frac{3}{2}} = M,$$

which cannot be integrated in an analytical way.

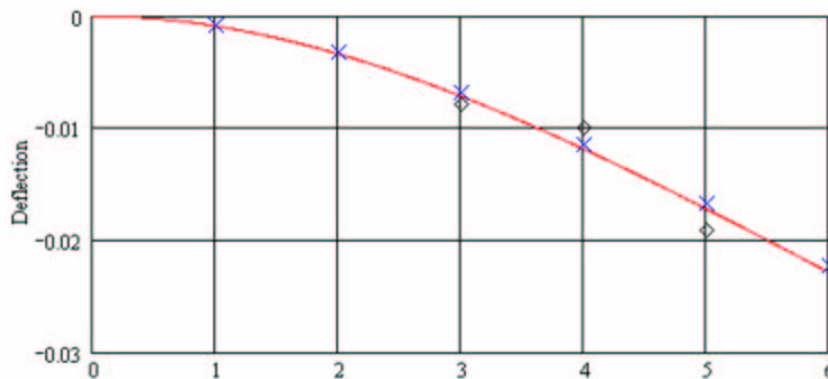
To solve the problem, we have chosen the finite difference method (in the interval  $[0, l]$ , the grid is chosen as  $x_i = i$ ,  $i = 1, 2, \dots, 6$ ) that reduces the solution of the linear differential equation (1) to the solution of a system of linear algebraic equations

$$\begin{cases} 4u_1 - u_2 = 0 \\ u_{i-1} - 2u_i + u_{i+1} = P(l - x_i)/EJ, \quad \text{for } i = \overline{2,5} \end{cases} \quad (6)$$

and an estimate of the accuracy of the resulting solution can be made by various measures, for example

$$\|y - u\| = \frac{1}{6} \sum_{i=1}^6 |y(x_i) - u_i|. \quad (7)$$





**Fig. 2.** Deflections of the beam: solid curve—analytical solution,  $x$ 's—numerical solution, dmnd's—experimental results

## 2.5 Estimation of the model accuracy

Adequacy of the model is most frequently estimated by comparing the experimental results with the numerical and/or analytical (if the later is possible) solutions, i.e., the norms are calculated

$$\|\bar{y} - u\| = \frac{1}{3} \sum_{i=3}^5 |\bar{y}_i - u_i| \quad \text{and/or} \quad \|y - \bar{y}\| = \frac{1}{3} \sum_{i=3}^5 |y(x_i) - \bar{y}_i|. \quad (8)$$

## 2.6 Numerical results

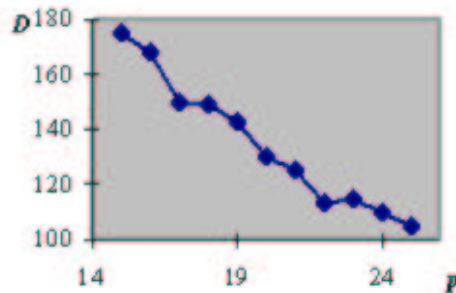
It is sensible to present the calculation results for a student, illustrated in Fig. 2 and Table 1, and norms (7), (8) equal to

$$\|y - u\| = 3.704 \cdot 10^{-4}, \quad \|\bar{y} - u\| = 8.715 \cdot 10^{-3}, \quad \|y - \bar{y}\| = 8.504 \cdot 10^{-3}.$$

**Table 1.** Analytical and numerical results

$x_i$	1	2	3	4	5	6
$y(x_i)$	0.00090	0.00339	0.00714	0.01185	0.01720	0.02286
$u_i$	0.00079	0.00317	0.00683	0.01143	0.01667	0.02222
$\bar{y}(x_i)$			0.00783	0.00991	0.01920	

In our opinion, it is impossible to attain the stated aims by merely listening to the general course of mathematics, though traditional mathematical courses ensure a student has the main body of mathematical knowledge. In order to attain all the above targets, cooperation between the lecturers of mathematics and engineering departments is necessary. This will initiate the appearance of special mathematical courses, such as modern methods of applied mathematics, special sections of pure



**Fig. 3.** The plot of the demand curve

and applied mathematics, etc. We have started to implement this. At the final stage of teaching of modern applied methods, students should follow special courses from their specialism in order to master the appropriate methods of applied mathematics and they should reflect these in their graduation theses.

### 3 A Problem from Economics

It is essential that engineers understand the fundamentals of economics. The following problem is a useful exercise for them. Demand for a commodity is given in a table.

**Table 2.** Price and demand

Price $p$	15	16	17	18	19	20	21	22	23	24	25
Demand $D$	175	168	150	149	143	130	125	113	115	110	105

Students are required to find at what price a monopolist can sell a good in order that his income is highest if it is known that the demand function for this good has the shape  $D = \exp(ap + b)$ .

For the primary stage of study of the problem a graph of the demand function is drawn:

The parameters  $a$  and  $b$  in the given relationship must be found using regression. A lecturer can allow students to decide for themselves which software package to use. This makes them recall the mathematical capabilities of packages. Engineers may make calculations of the parameters of such a relationship using computer algebra packages, while economists may use statistical packages. However, for both of them one can recommend finding the relationship with *Microsoft Excel* by applying the least squares method (LSM). Certainly, using Excel to find the regression equation takes more time, but this demonstrates clearly the essence of LSM and its universality.

The construction of the model and its study is comprised of the following steps:

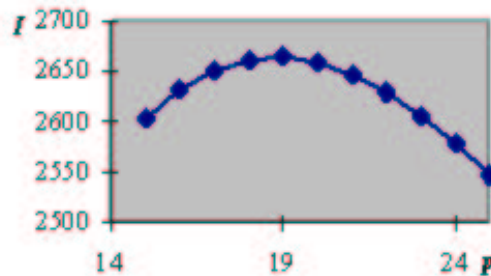


Fig. 4. The plot of the income curve

1. Finding the parameters  $a$  and  $b$  of the regression function  $D = \exp(ap + b)$  by applying LSM.
2. Constructing the function  $I = pD$  and finding its maxima.
3. Interpretation of the results in terms of economics.

Using Excel, we obtain that  $D = \exp(0.053p + 5.955)$ . We can analyze the expression  $I = pD$  with unknown  $a$  and  $b$ , however, using the expression with numerical coefficients, we can draw its graph by computer. This enables us to make a primary analysis of the income  $I$ . In this case, the graph of income shows that  $I$  really has a maximum. However, this also demonstrates the inaccuracy of graphical methods, because it is impossible to find the maximum point from the graph accurately enough. Therefore one has to apply mathematical analysis methods to the function  $I$ .

Here we may recommend use of a computer algebra package (it is simpler). In the lecture-room the students not only do that, but also discuss why it is better to do so.

The identification of a stationary point of the function  $I$  by the package *Mathematica* comprises only a few lines:

```
a=-0.053
b=5.9547
i=p*Exp[a*p+b]
deriv=D[i, p]
FindRoot[deriv=0, {p,0}]
```

The price found is  $p_0 = 18.8766$ , and the income is  $I = 2676.23$ .

While considering this solution in terms of economics, it is worth explaining why for  $p > p_0$  the income starts diminishing, i.e., the growth in price does not compensate the decrease in demand. This conclusion mathematically grounds the result and explains to the students of economics the truths they already know from their course of economics. Meanwhile, to engineers, this can be presented as a method for acquiring some knowledge of economics or investigating economic phenomena.

The problem is completed by stressing all the stages used for problem analysis and mathematical modeling of a real situation. This allows students to appreciate the role of mathematics and computer package application as a tool for solving a problem.

However, not every engineer will have access to an expensive mathematical package after graduating from the university. For example, it may be cheaper and quicker for a designer or an economist, working with a general packages (of *CAD* or *Excel* type), to write a supplement to the program rather than to acquire a mathematical package. This can be a *macro* collection for the program *Excel* or *add-on* in some language for a designing package. To do this requires knowledge of the problem solution algorithm.

## 4 Conclusions

1. Applying mathematical modeling in a simplified way to the simulation of a real situation frequently requires the application of several mathematical methods. In order to choose the appropriate methods to investigate the model, it is necessary to have a knowledge of the capabilities of mathematical and other packages.
2. Computer applications may be reasonable for a primary investigation of the given problem before constructing a mathematical model.
3. Students of lower courses may have insufficient knowledge of the matter, therefore in the first year the samples of models ought to be carefully chosen. This obliges a lecturer of mathematics to find out which topics the students have studied in the lectures of other subjects. In our opinion, it is easier for lecturers to do this at small (branches of) universities, which have the advantage of closer cooperation between the lecturers of different departments and faculties, than those at large ones.
4. Accumulation of the necessary mathematical knowledge has to be performed consistently and in various ways, by using computers from the very first year.

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### About the authors

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# Understanding of Mathematics is Prerequisite for Efficient Use of Computers

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In the last fifteen years we have frequently encountered the opinion that, thanks to the new technologies, we need to learn mathematics only to the very basic level. According to this extreme opinion sufficient understanding of mathematics is such as to enable one to realize three main steps: to formulate an engineering problem in mathematical language, to perform suitable commands into a computer (input), and to read the needed information from the obtained output. Some schools follow such recommendations, without any critical appraisal, perhaps motivated by the reduced teaching loads.

When doing just basic mathematics we do not need deep understanding of mathematics subject. However, as tasks become more complicated, deeper understanding is needed. First of all, we must prepare a suitable input, if we wish the computer to work properly. After the computer's work is finished, we have to simplify or to control the obtained results. This last work is extremely important when dealing with demanding task. The reason is that computer can give a wrong result, even without any warning. It seems that the most important finding is the fact that we must control the computer itself. However, to control the computer we need a fairly good understanding of mathematics. We shall illustrate this assertion with some instructive examples, which show the vital interplay between mathematics and computers. The first three examples are relatively simple, but the last one is quite complicated.

**Example 1.** Using *Mathematica* [2] and sequences  $f(n)$ ,  $\varphi(n)$ ,  $f^*(n)$ ,  $g(n)$ , and

$\gamma(n)$ , defined recursively as:

1.  $f(0) := \pi, \quad f(n) := \sqrt{f(n-1)}$
2.  $\varphi(0) := 3.14, \quad \varphi(n) := \sqrt{\varphi(n-1)}$
3.  $f^*(0) := \frac{314}{100}, \quad f^*(n) := \sqrt{f^*(n-1)}$
4.  $g(0) := \frac{1}{10}, \quad g(n) := \sqrt{g(n-1)}$
5.  $\gamma(0) := 0.1, \quad \gamma(n) := \sqrt{\gamma(n-1)},$

we produce the following table

$n$	25	30	35	40	45
$(f(n))^{2^n}$	$\pi$	$\pi$	$\pi$	$\pi$	$\pi$
$(\varphi(n))^{2^n}$	3.1400	3.1400	3.1400	3.1402	3.1287
$(f^*(n))^{2^n}$	$\frac{157}{50}$	$\frac{157}{50}$	$\frac{157}{50}$	$\frac{157}{50}$	$\frac{157}{50}$
$(g(n))^{2^n}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$(\gamma(n))^{2^n}$	0.1	0.1	0.1	0.1	0.0998

$n$	50	55	60	65
$(f(n))^{2^n}$	$\pi$	$\pi$	$\pi$	$\pi$
$(\varphi(n))^{2^n}$	2.7183	1	1	1
$(f^*(n))^{2^n}$	$\frac{157}{50}$	$\frac{157}{50}$	$\frac{157}{50}$	$\frac{157}{50}$
$(g(n))^{2^n}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$(\gamma(n))^{2^n}$	0.1054	1	1	1

The second, the fourth, and the fifth row confirm that the operation of sequentially extracting square root  $n$ -times is the inverse to the operation of  $n$ -times squaring. But how to explain the third and the sixth row, where the rows end with units instead of the initial number? To do this, we must understand that *Mathematica* produced the second, the fourth, and the fifth rows using symbolic calculations, while the other two rows were produced numerically using finite arithmetic. Considering these two rows we must understand that sequence  $n \mapsto x^{\frac{1}{2^n}}$  converges to 1 increasingly/decreasingly for  $0 < x < 1$  and  $x > 1$ , respectively.

**Example 2.** Searching for the indefinite integral  $\int \ln(1-x) dx$ , i.e. searching for real function  $f(x)$  of real variable, defined on interval  $(-\infty, 1)$ , such that  $f'(x) \equiv \ln(1-x)$  for  $x < 1$ , *Mathematica* gives definitely wrong answer producing the output

$$\int \ln(1-x) dx \rightarrow -x + \ln(1-x) - \ln(-1+x).$$

This is so because for real  $x < 1$  the function on the right side is not real at all. What should we do in the case above when we do not know the method of integration

by parts, but we are familiar with the substitution method in the definite integral? In this case we can solve the problem. Indeed, because we can directly confirm *Mathematica's* output

$$\int \ln x \, dx \rightarrow x(\ln x - 1)$$

on the interval  $(0, \infty)$ , we use substitution  $1 - x = t$  to find

$$\int \ln(1 - x) \, dx = - \int \ln t \, dt = (x - 1) [\ln(1 - x) - 1] + \text{const.}$$

for  $x < 1$ . We see that without well-grounded understanding of mathematics we have no idea how to proceed in the example discussed above.

**Example 3.** Dealing with expression<sup>1</sup>

$$B(a, m, n) := \binom{a}{m} \cdot \frac{m^m (a - m)^{a-m}}{n^n (a - n)^{a-n}} \cdot \sqrt{\frac{m(a - m)}{n(a - n)}} \cdot \exp \left[ \frac{a}{12} \left( \frac{1}{m(a - m)} - \frac{1}{n(a - n)} \right) \right], \quad (3.1)$$

we use substitutions

$$d(a, n) := \frac{n - \frac{a}{2}}{\frac{a}{2}} = 2 \frac{n}{a} - 1, \quad (3.2)$$

$$\varphi(t) := (1 + t)^{1+t} \quad (3.3)$$

and

$$D(a, t) := \sqrt{1 - t^2} \cdot [\varphi(t) \varphi(-t)]^{a/2} \cdot \exp \left( \frac{1}{3a(1 - t^2)} \right), \quad (3.4)$$

to obtain compact expression

$$B(a, m, n) = \binom{a}{m} \cdot \frac{D(a, x)}{D(a, y)}, \quad (3.5)$$

where  $x = d(a, m)$ ,  $y = d(a, n)$ . We would like to examine if (3.1) is really equivalent to (3.5), i.e. we would like to check if we did not make some error during our calculations. Hence, one dealing with difference

$$\Delta(a, m, n) := \binom{a}{m} \cdot \frac{D(a, x)}{D(a, y)} - B(a, m, n),$$

using substitutions (3.2)–(3.4). Unfortunately, using command *FullSimplify* for symbolic simplification we do not succeed to obtain  $\Delta(a, m, n) \equiv 0$ . After this disappointment we check equality  $\Delta(a, m, n) = 0$  for some special values of  $a$ ,  $m$  and  $n$ . We are disappointed again with the result  $|\Delta(100\pi, 10, 170n)| > 5.8 \times 10^{79}$ , obtained using the default precision of numerical calculations. Fortunately,  $|\Delta|$  is

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<sup>1</sup>which is close to  $\binom{a}{n}$  for integers  $m$  and  $n$ , such that  $2 \leq m \leq n < a$



relatively small as regards  $|B|$ . Indeed, using *Mathematica* [2], we directly estimate  $B(100\pi, 10, 170) > 5.8 \times 10^{92}$ . Now, checking the relative difference as

$$\left| \frac{\Delta(100\pi, 10, 170)}{B(100\pi, 10, 170)} \right| < 1.1 \times 10^{-13} \approx 0,$$

we are quite satisfied. Moreover, increasing the precision of numerical calculations, using command `Block[MaxExtraPrecision = 200, Δ(100π, 10, 170)]`, we obtain the output

$$|\Delta(100\pi, 10, 170)| \rightarrow 0. \times 10^{-204}$$

which indicates that  $\Delta(100\pi, 10, 170)$  should be equal to 0. Fortunately, results similar to the one above are obtained also using some other special values of  $a$ ,  $m$  and  $n$ . Consequently, we conclude that expressions (3.1) and (3.5) are equivalent using substitutions (3.2)–(3.4).

**Example 4.** Let us examine the series  $\sum_{k=1}^{\infty} (\sin \sqrt{k})/k$  and  $\sum_{k=1}^{\infty} (\sin \sqrt{k})/\sqrt{k}$ . We are interested whether they converge, and if they do, what are the numerical values of their sums. Although *Mathematica* [2] can symbolically find that  $\sum_{k=1}^{\infty} (\sin k)/k = (\pi - 1)/2$ , it does not succeed in answering our questions in this way. Therefore we try to put our questions differently—numerically. Asking for the numerical values of the infinite sums above, using command `NSum`, we obtain the following output

$$NSum[Sin[\sqrt{k}]/k, \{k, 1, \infty\}] \longrightarrow 1.5307 \quad (4.1)$$

$$NSum[Sin[\sqrt{k}]/\sqrt{k}, \{k, 1, \infty\}] \longrightarrow -0.270509. \quad (4.2)$$

However, we are somewhat suspicious due to our previous experiences. Therefore, using *Mathematica* [2], we compute the lists of partial sums for both series. Figures 1 and 2 shows the sequence of partial sums of these series.

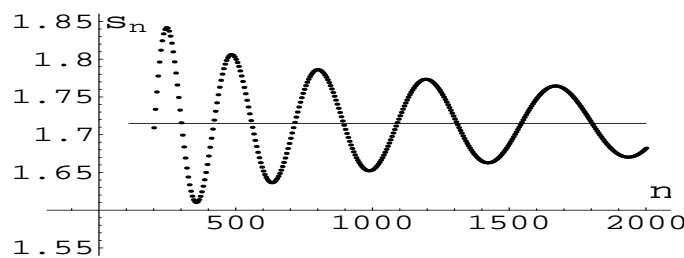


Figure 1: Convergence of partial sums  $s_n = \sum_{k=1}^n (\sin \sqrt{k})/k$ .

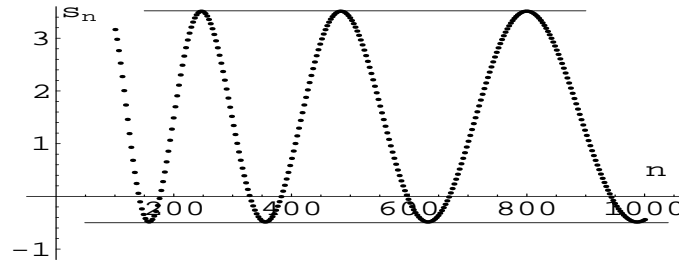


Figure 2: Convergence of partial sums  $s_n = \sum_{k=1}^n (\sin \sqrt{k})/\sqrt{k}$ .

The first figure suggests convergence of the first series, however the second figure raises some doubt about the convergence of the second series. In this uncertain situation we need the help of mathematics. Fortunately, we have at our disposal Theorem 2 [1, p. 119], which we quote here as

**THEOREM.** *If  $f \in C^p[1, \infty)$ ,  $\int_1^\infty |f^{(p)}(x)| dx$  converges, and finite limits  $\lambda_0 := \lim_{n \in \mathbb{N}} f(n)$  and  $\lambda_k := \lim_{n \rightarrow \infty} f^{(k)}(n)$  exist for all positive integers  $k \leq p - 1$ , then*

- (i) *The series  $\sum_{k=1}^\infty f(k)$  converges.  $\iff$  The sequence  $n \mapsto \int_1^n f(x) dx$  converges.*
- (ii) *If the series  $\sum_{k=1}^\infty f(k)$  converges to  $s$ , then  $\lambda_0 = 0$  and*

$$s = \gamma + \lim_{\substack{n \in \mathbb{N} \\ n \rightarrow \infty}} \int_m^n f(x) dx,$$

$\gamma$  being Euler's constant of a function  $f$ , and there holds the equality

$$s = \sum_{k=1}^{m-1} f(k) + \int_m^\infty f(x) dx + [\sigma_p(\infty) - \sigma_p(m)] + \delta_p(m),$$

where

$$\sigma_p(m) = \sum_{j=1}^p \frac{B_j}{j!} f^{(j-1)}(m), \quad \sigma_p(\infty) = \sum_{j=1}^p \frac{B_j}{j!} \lambda_{j-1},$$

$B_j = B_j(0)$  being Bernoulli coefficients,  $B_j(x)$  Bernoulli polynomials,  $B_1 = -\frac{1}{2}$ ,  $B_2 = \frac{1}{6}$ ,  $B_3 = B_5 = B_7 = 0$ ,  $B_4 = B_8 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}, \dots$ , and

$$|\delta_p(m)| \leq \frac{\mu_p}{p!} \int_m^\infty |f^{(p)}(x)| dx, \quad \mu_p = \max_{0 \leq x \leq 1} |B_p(x)|$$

with  $\mu_1 = \frac{1}{2}$ ,  $\mu_2 = \frac{1}{6}$ ,  $\mu_4 = \mu_8 = \frac{1}{30}$ ,  $\mu_6 = \frac{1}{42}$ ,  $\mu_3 < \frac{1}{20}$ ,  $\mu_5 = \frac{1}{35}$ , and  $\mu_7 < \frac{1}{30}$ .

**Question 1.** Does the series  $\sum_{k=1}^\infty \frac{\sin \sqrt{k}}{k}$  converge?

This question can actually be solved by putting  $p = 1$  in the *Theorem* above. Here we are dealing with function  $f: [1, \infty) \rightarrow \mathbb{R}$ ,  $f(x) \equiv \frac{\sin \sqrt{x}}{x}$ , having  $\lambda_0 = \lim_{n \rightarrow \infty} f(n) =$

0 and the integral convergent. Indeed, substituting  $x = t^2$ , we reduce our integral to Diriclet's one [3]:

$$\int_1^\infty \frac{\sin \sqrt{x}}{x} dx = 2 \int_1^\infty \frac{\sin t}{t} dt = \pi - 2 \int_0^1 \frac{\sin t}{t} dt, \quad (4.3)$$

where the last integral does not cause difficulties. Because

$$f'(x) \equiv \frac{\sqrt{x} \cos \sqrt{x} - 2 \sin \sqrt{x}}{2x^2}, \quad (4.4)$$

integral  $\int_1^\infty |f'(x)| dx$  converges. Hence, from the preceding *Theorem* it follows that the series  $\sum_{k=1}^\infty (\sin \sqrt{k})/k$  converges too.

**Question 2.** What is the sum  $s$  of the series  $\sum_{k=1}^\infty \frac{\sin \sqrt{k}}{k}$ ?

We shall use the above *Theorem (ii)*, setting  $f(x) \equiv \frac{\sin \sqrt{x}}{x}$  and  $p = 3$ . Because  $\lambda_0 = \lim_{n \rightarrow \infty} f(n) = 0$ , and according to (4.4), also  $\lambda_1 = \lim_{n \rightarrow \infty} f'(n) = 0$ , we have  $\sigma_3(\infty) = 0$ . Consequently we obtain

$$s = \sum_{k=1}^{m-1} f(k) + \int_m^\infty f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \delta_3(m), \quad (4.5)$$

where

$$|\delta_3(m)| \leq \frac{1}{120} \int_m^\infty |f'''(x)| dx \quad (4.6)$$

Using the substitution  $x = t^2$  first and then the method of integration by parts, we transform the integral as

$$\begin{aligned} \int_m^\infty f(x) dx &= 2 \int_{\sqrt{m}}^\infty \frac{\sin t}{t} dt \\ &= \frac{2 \cos \sqrt{m}}{\sqrt{m}} + \frac{2 \sin \sqrt{m}}{m} - \frac{4 \cos \sqrt{m}}{m\sqrt{m}} - \frac{12 \sin \sqrt{m}}{m^2} + \rho(m), \end{aligned} \quad (4.7)$$

where the remainder

$$\rho(m) = -12 \int_{\sqrt{m}}^\infty (\sin t) \frac{d}{dt} (t^{-4}) dt$$

is estimated as

$$|\rho(m)| \leq 12 \int_{\sqrt{m}}^\infty \left| \frac{d}{dt} (t^{-4}) \right| dt = \frac{12}{m^2}. \quad (4.8)$$

Considering (4.4) we estimate

$$|f'''(x)| \leq \frac{1}{8} x^{-5/2} + \frac{9}{8} x^{-3} + \frac{33}{8} x^{-7/2} + 6 x^{-4},$$

for  $x > 0$ . Thus, due to (4.6), there follow relations

$$\begin{aligned} |\delta_3(m)| &\leq \frac{1}{120} \int_m^\infty \left( \frac{1}{8} x^{-5/2} + \frac{9}{8} x^{-3} + \frac{33}{8} x^{-7/2} + 6 x^{-4} \right) dx \\ &= \frac{1}{120} \left( \frac{1}{12m\sqrt{m}} + \frac{9}{16m^2} + \frac{33}{20m^2\sqrt{m}} + \frac{2}{m^3} \right). \end{aligned} \quad (4.9)$$

Consequently, concerning (4.5), (4.4) and (4.7), we conclude with expression

$$s = S(m) + \Delta(m), \quad (4.10)$$

where

$$S(m) = \sum_{k=1}^{m-1} \frac{\sin \sqrt{k}}{k} + \frac{2 \cos \sqrt{m}}{\sqrt{m}} + \frac{5 \sin \sqrt{m}}{2m} - \frac{97 \cos \sqrt{m}}{24m\sqrt{m}} - \frac{143 \sin \sqrt{m}}{12m^2} \quad (4.11)$$

and where, due to (4.8) and (4.9), there hold the estimates

$$\begin{aligned} |\Delta(m)| &= |\rho(m) + \delta_3(m)| \\ &\leq \frac{1}{1440 m\sqrt{m}} + \frac{7683}{640 m^2} + \frac{11}{800 m^2\sqrt{m}} + \frac{1}{60 m^3} \end{aligned} \quad (4.12)$$

for integer  $m \geq 1$ .

Using *Mathematica* [2] we find  $S(170) = 1.71555\dots$  and  $|\Delta(170)| < 4.2 \times 10^{-4}$ , due to formula (4.11) and the estimate (4.12). Consequently we obtain for the sum  $s$  of our series the following estimate:  $1.71513 = 1.71555 - 0.00042 < s < 1.71556 + 0.00042 = 1.71598$ . Hence,  $s = 1.715\dots$ , i.e. the sum  $s$  is evaluated to three decimal places accurately with the absolute value of relative error less than 0.06%. In this way we again confirmed that the computer result was wrong.

In connection with this computation we wish to emphasize that partial sum  $s_{170} = \sum_{k=1}^{170} \frac{\sin \sqrt{k}}{k} = 1.5767\dots$  poorly approximates  $s$ ; the absolute value of relative error of this approximation is greater than 8%. Anyhow, it is evident that the computation of the sum  $s$  using only computer is not sufficient.

**Question 3.** Does the series  $\sum_{k=1}^\infty \frac{\sin \sqrt{k}}{\sqrt{k}}$  really diverge?

To answer this question we shall use our *Theorem* above setting  $p = 2$  for integrand  $f: [1, \infty) \rightarrow \mathbb{R}$ ,  $f(x) \equiv \frac{\sin \sqrt{x}}{\sqrt{x}}$ , which has the derivatives

$$f'(x) \equiv \frac{\cos \sqrt{x}}{2x} - \frac{\sin \sqrt{x}}{2x^{3/2}} \quad (4.13)$$

and

$$f''(x) \equiv -\frac{\sin \sqrt{x}}{4x^{3/2}} - \frac{3 \cos \sqrt{x}}{4x^2} + \frac{3 \sin \sqrt{x}}{4x^{5/2}}. \quad (4.14)$$

Thus  $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} f'(n) = 0$  and, by (4.13), integral  $\int_1^\infty |f^{(2)}(x)| dx$  converges. Hence, we have only to examine the sequence  $n \mapsto \int_1^n f(x) dx$ .

Substituting  $x = t^2$  we obtain equalities

$$\int_m^n \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int_{\sqrt{m}}^{\sqrt{n}} \frac{\sin |t|}{|t|} 2t dt = 2 \int_{\sqrt{m}}^{\sqrt{n}} \sin t dt = 2 (\cos \sqrt{m} - \cos \sqrt{n}) \quad (4.15)$$

for  $m, n \in \mathbb{N}$ .

Therefore, the sequence  $n \mapsto \int_1^\infty f(x) dx = 2 (\cos 1 - \cos \sqrt{n})$  diverges due to the *Lemma*<sup>2</sup> below. According to the preceding *Theorem* we conclude that the series  $\sum_{k=1}^\infty (\sin \sqrt{k})/\sqrt{k}$  diverges as well.

LEMMA. *Every point of the closed interval  $[-1, 1]$  is a limit point of the sequences  $n \mapsto \cos n$  and  $n \mapsto \sin n$ .*

**Remark.** We have proved divergence of the series  $\sum_{k=1}^\infty (\sin \sqrt{k})/\sqrt{k}$ , however we do not know the numerical behavior of its partial sums. Fortunately, even this question can be answered adequately. Indeed, by [1, p. 118], items (23a) and (23b), we have

$$\sum_{k=m}^n f(k) = \int_m^n f(x) dx + \frac{1}{2} [f(m) + f(n)] + \frac{1}{12} [f'(n) - f'(m)] + \rho_3(m, n) \quad (4.16)$$

where the remainder  $\rho_3$  is estimated as

$$|\rho_3(m, n)| \leq \frac{1}{120} \cdot \int_m^n |f'''(x)| dx. \quad (4.17)$$

Using these relations we obtain for the  $n$ th partial sum  $s(n) := \sum_{k=1}^n \frac{\sin \sqrt{k}}{\sqrt{k}}$  the formula

$$s(n) = 1.5138 - 2 \cos \sqrt{n} + \frac{\sin \sqrt{n}}{2\sqrt{n}} + \frac{\cos \sqrt{n}}{24n} - \frac{\sin \sqrt{n}}{24n^{3/2}} + \delta(n),$$

where

$$0 < \delta(n) < 1.6 \times 10^{-4}$$

for  $n \geq 30$ . However,

$$s(n) = \tilde{s}(n) - 0.0238 + \delta_0(n),$$

where  $0 < \delta_0(n) < 0.05$  for every positive integer  $n$  and

$$\tilde{s}(x) := 1.5138 - 2 \cos \sqrt{x} + \frac{\sin \sqrt{x}}{2\sqrt{x}} + \frac{\cos \sqrt{x}}{24x} - \frac{\sin \sqrt{x}}{24x^{3/2}}.$$

Figure 3 shows the graph of sequence  $s(n)$  as well as the graph of function  $\tilde{s}(x)$  of continuous variable  $x$ .

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<sup>2</sup>well known in the theory of dynamic systems

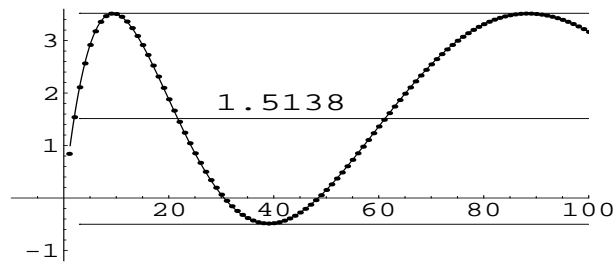


Figure 3: Sequence  $s(n)$  and its continuous approximation  $\tilde{s}(x)$ .

**Conclusion.** We have given, in our opinion, clear example of the vital interplay between mathematics and computers. We have illustrated the fundamental truth that mathematics and computers do not exclude each other but they complement one another. Moreover, we have demonstrated, using the examples above, that for solving non-elementary tasks a good hardware alone is not sufficient. In addition a solid knowledge of mathematics is required.

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# Supporting the Transition from School Mathematics to University Mathematics

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## Abstract

For many years the mathematics community in Higher Education in the United Kingdom has been expressing concern about the decreasing level of mathematical competence of students on entry to university. In this paper we outline some objective evidence of this decline, based on results of a diagnostic test used in the first week of students' university career. We discuss one measure several universities have introduced to address the problems caused by the change in students' entry competences, namely mathematics support centres. Finally, we outline two projects **mathcentre** and **mathtutor** providing resources that all universities may use to help students make the transition from school to university mathematics.

## Introduction

For several years the higher education mathematics community in England has been raising concerns about the competence of new undergraduates in disciplines with a high mathematical content. A series of reports, published by various professional bodies [1–4], has added credibility to these concerns. In 2003 the Government set up a national inquiry under the chairmanship of Prof Adrian Smith. The report of this inquiry, 'Making Mathematics Count' [5], was published earlier this year.

'Making Mathematics Count' contains a series of recommendations about mathematics education at a range of levels and urges university mathematics departments to play more active and co-ordinated roles in promoting mathematics in schools. The report also recommends the establishment of a National Centre for Excellence in Mathematics Teaching to co-ordinate the activities of nine Regional Mathematics Centres. In order for these to be set up and play the role Prof Smith envisages a substantial sum of money (in excess of £150 million) is required. Up to now the Government has not made an official response to the report. However, Charles Clarke, the Minister for Education and Skills, did say in a presentation to the Heads of Departments of Mathematical Sciences conference [6]:

*'We profoundly welcome the report of Professor Adrian Smith, we are delighted that he has published his report so clearly and we are very keen to move to implement the recommendations he has set out.'*

Even if the Government does respond positively to the recommendations of the Smith report it will be several years before the changes proposed will have a major effect on the mathematical competences of new undergraduates. In the meantime higher education must put in place strategies to support students at the transition from school to university mathematics.

## Changes During the 1990s

Diagnostic testing in mathematics for new undergraduates was introduced on a large scale at Coventry University in 1991. A 50 question multiple choice test covering arithmetic, algebra, lines and curves, trigonometry and basic calculus is used. The test has remained unchanged and so data from student attempts at the test since 1991 give an objective measure of how entry competences have changed amongst students with the same entry qualifications. In this paper we present a short summary of the most important changes observed over the period 1991 to 2001. A fuller discussion can be found elsewhere [7].

Scores achieved in the diagnostic test dropped considerably over this period for students with the same mathematics entry qualification. In 1991 the average score of those students with A level mathematics grade D was 37.3 (out of 50). By 2001 this had fallen to 29.1. The topics where there was the greatest decline in competency were primarily in algebra—the percentage of the cohort correctly answers questions relating to factorising quadratic expressions, solving quadratic equations and handling fractional indices fell by over 35 points.

A level mathematics has pass grades A to E and two fail grades N (for near miss) and U for unclassified. In 1991 the average score in the diagnostic test of students with grade N was 34.4. In 2001 the average score of students with grade B was 33.8.

## Mathematics Support Centres

The changes discussed in the last section are well-known to the higher education mathematics community. One strategy that has been adopted in an attempt to reduce the problems caused by this decline in entry competences is the establishment of mathematics support centres [8,9]. A survey carried out in 2001 [10] indicated that almost 50% of UK universities had some kind of mathematics support provision and since that time other universities have started to offer this type of facility. Although there is variation in detail, most mathematics support centres offer similar services. They provide support in addition to that given in the student's normal diet of lectures and tutorials. This support includes access to a range of resources; the most popular of which are short, focused leaflets. In addition there is usually the opportunity to receive one-to-one help from a tutor.

Mathematics support centres at most universities are offering assistance in the same areas, usually algebraic manipulation, trigonometry and calculus. This commonality of topics of concern led to a project to establish a virtual support centre offering high quality resources to both students and staff. Funding was secured



from the Learning and Teaching Support Network (LTSN) and *mathcentre* [11] was established.

In parallel to this development, a complementary project was successfully submitted to HEFCE's fourth round of bidding to the Fund for the Development of Teaching and Learning (FDTL). The aim of this project was to develop electronic materials to support students at the school/university interface. The central focus of these materials is a collection of e-tutorials covering important mathematical principles and techniques. These resources are to be delivered on CD-ROM and DVD-ROM under the title of *mathtutor*.

## Mathcentre and Mathtutor

Although begun as two separate projects, *mathcentre* and *mathtutor* have now become inextricably linked. A common approach to resource development has been adopted. For *mathtutor*, the e-tutorials are the central resource. These are videos of high quality teachers presenting detailed coverage of a range of mathematical topics. Supporting these videos are a range of other materials including diagnostic tests, e-texts (suitable for printing) that give parallel coverage (what might be called pseudo-transcripts) to the videos, summary leaflets and interactive exercises.

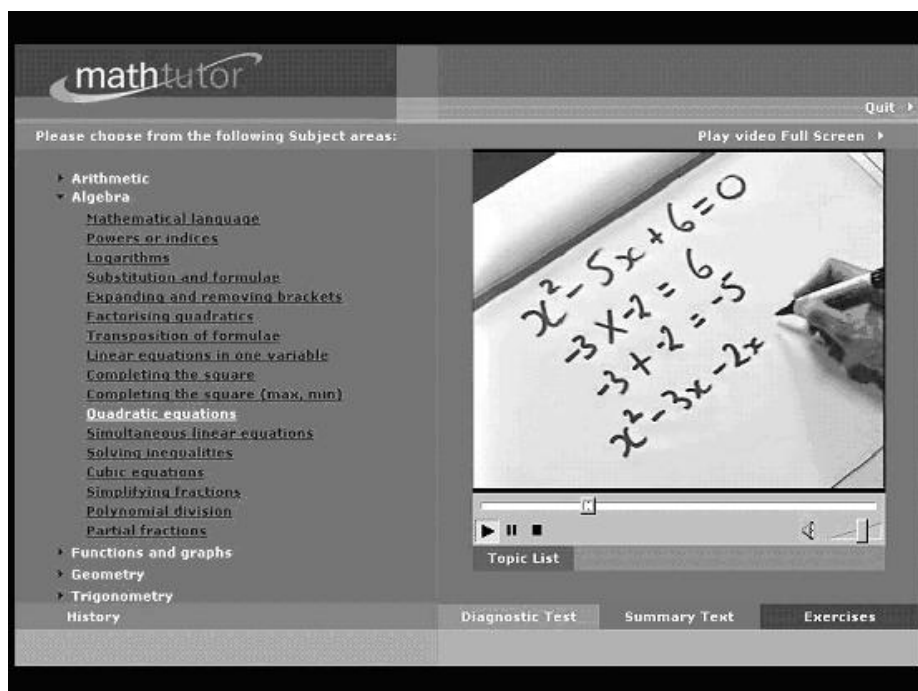


Figure 1: *mathtutor* displaying an e-tutorial

Figure 1 shows the mathtutor interface with the e-tutorial on quadratic equations running. The other resources can be accessed using the buttons underneath the screen.

The five question diagnostic tests align with the videos in such a way that if for example a student answers the first three questions correctly but the last two incorrectly they can jump straight to the appropriate point of the video, automatically skipping over the presentation of those parts of the topic in which they have demonstrated competence.

The videos are typically around 30 minutes long. It was originally envisaged that these materials would not be provided by the mathcentre web-site because of the quality that could be expected. However recent developments in video-streaming have now made it possible for the videos to be delivered over the web and, with a broadband connection, the quality is good.

The **mathcentre** web-site is designed for both students and staff. Figure 2 shows the entry point to the student section. As a student you can specify the discipline you are studying and then you will be offered resources appropriate to that discipline. There is a global search facility so that if the topic you are interested in is not on the menu in your discipline you can still find materials on that topic.

The staff section of the web-site contains bundles of resources to save staff from having to download them individually and also other resources such as good practice guides, LTSN occasional reports and the MathsTEAM booklets [12–14].

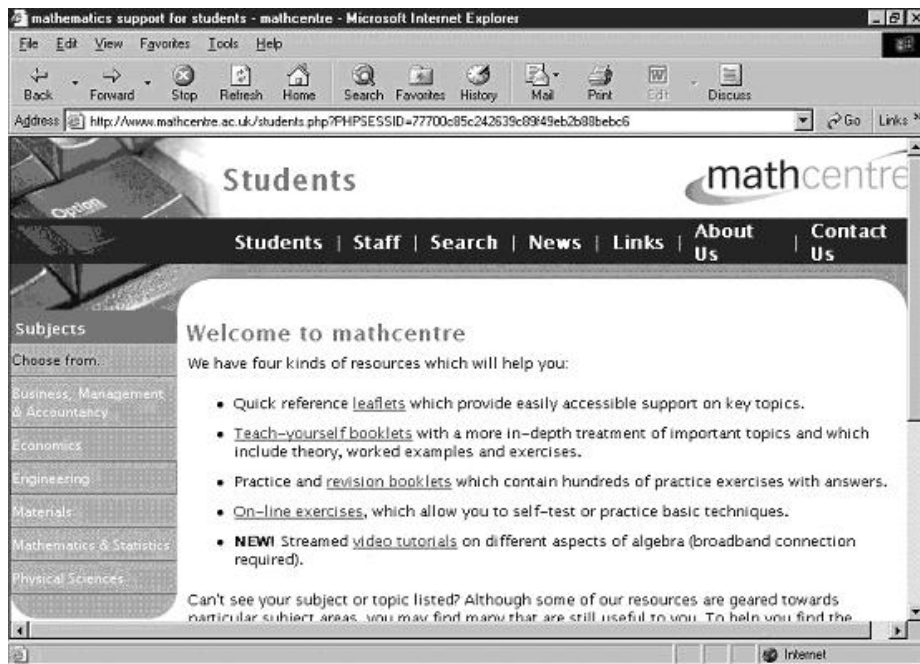


Figure 2: The student section of the mathcentre web-site

## Evaluation and the Future

The projects *mathcentre* and *mathtutor* are both on-going. The first topic to be covered was algebra and a pilot disk was produced in 2003. A systematic evaluation of this disk was undertaken and the findings were very positive. Several hundred users of the disk were asked to complete a short questionnaire rating the different resources on the disk and giving some overall opinions. The results from this questionnaire are shown in Tables 1 and 2 below.

	Very useful	Useful	OK	Not Useful
Diagnostic	23	40	26	11
e-tutorial	28	36	24	12
Text	22	37	33	8
Exercises	41	33	20	6

**Table 1:** Opinions of the various resources on the *mathtutor* disk

	Agree	Disagree
I would like to see more	85	15
I like this sort of thing	81	19
I found the disk easy to navigate	92	8

**Table 2:** Overall opinions of the *mathtutor* disk

By far the most valued resource on the disk is the interactive exercises—this echoes findings about the web-site for the Mathematics Support Centre at Coventry University [15], where the most used resource is the on-line tests. The percentage of respondents indicating that they found any of the resources not useful was encouragingly low.

Based on the feedback received from this evaluation small modifications are being made to interface. The schedule is now to release in September 2004 a revised algebra disk and further disks covering co-ordinate geometry, trigonometry, differentiation and integration. At a later date, disks covering arithmetic, functions, sequences and series, vectors and introductory differential equations will be produced.

A formal evaluation of the **mathcentre** web-site has not yet been undertaken. Evidence of the value of the site is the number of hits it receives (over 500 per day and increasing) and the unsolicited emails sent praising the site and the resources it offers. Up to now publicity for the site has been relatively low key. It is intended to advertise **mathcentre** much more actively with new students in September/October 2004 and it is expected that this will lead to even higher usage of the site. The number of resources available on the **mathcentre** site will also increase as more resources are prepared for the **mathtutor** project. In addition, the number of discipline categories by which students can identify themselves to the web-site will also be increased as more, appropriate resources (for example, for nursing students) are made available.

## Conclusion

Both **mathcentre** and **mathtutor** have already proved their worth as support resources for students at the school/university interface. The volume of resources available will increase significantly in the near future thereby greatly increasing the value of these two projects.

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# The Concept of Elasticity in Teaching and Applications of Engineering Mathematics

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## Abstract

The paper points the concept of the elasticity of a function. Applications in decision making in economy are presented and impacts on instructing of engineering mathematics are discussed.

## 1 Elasticity

### Motivation

When investigating the dynamics of processes, current courses of engineering mathematics offer a standard tool—the notion of a derivative. In teaching calculus of functions as a part of engineering mathematics, the concept of a derivative of a function plays a key role. In practise, the concept is fully formalized, mathematically “treated” and also interpreted. The interpretation (in fact substituting the motivation-description of dynamics) has usually no priority (sometimes due to lack of time), some work is left to the related engineering disciplines. In this approach, concentrating only to a derivative, important aspects of the character of dynamics may be lost. Modelling of dynamics means to “summarize” the way in which changes in one variable affect some other variable. If the nature of the affect (or the response) is the rate of change, then the concept of a derivative fits and a *derivative* of a function  $y = f(x)$  is employed as a measure of *the instantaneous (rate of) change of  $y$  with respect to  $x$* , or equivalently, it measures the approximate change in  $y (= f(x))$  as a response to a unit change in  $x$ , i.e.  $f'(x) \approx f(x+1) - f(x)$ . But in a number of real situations, the concept of a derivative is not (directly) convenient to describe the character of responsiveness under consideration. Such situation is typical when modelling economic processes. For example, an economist might be interested to measure how the change in the price of a product affects the quantity demanded by the market. One problem that arises in the construction of such summary measures is that quite often variables  $x$  and  $y$  are not measured in the same units. For example, the quantity of potatoes is measured in kilos, the price of potatoes (per kilo) is measured in euros. We might then speak of an increase of 10 cents per kilo in the price of potatoes, leading to a fall of 2000 kilos of potatoes purchases. Similarly, we could speak of a fall in the price of apples by 10 cents per

dozen, leading to an increase in apple purchases of 1500 dozen. Now the question is whether potatoes are more or less responsive to price changes than are apples. The problem exists because the goods are measured in different units. To answer this question, economists (firstly Cournot in 1838, see Cournot (1897)) have developed the concept of *elasticity*. While a derivative is based on (absolute) changes in  $x$  or  $y$  respectively, elasticity is based on proportional changes in  $x$  or  $y$  respectively.

### Elasticity of a function

For a function  $y = f(x)$  denote by  $\Delta x$  **the change** in  $x$ . If  $x$  changes from  $x$  to  $(x + \Delta x)$ , **the proportional change** in  $x$  is  $\Delta x/x$  and **the proportional change** in  $y$  is

$$\frac{\Delta y}{y} = \frac{f(x + \Delta x) - f(x)}{f(x)}. \quad (1.1)$$

Then **the average (rate of) proportional change in  $y$  per unit proportional change** in  $x$  is  $(\Delta y/y)/(\Delta x/x)$ , i.e.

$$\frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{x}{f(x)} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (1.2)$$

The **elasticity** of the function  $y = f(x)$  is the function  $E_f(x)$  given by

$$E_f(x) = \lim_{\Delta x \rightarrow 0} \frac{x}{f(x)} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{x}{f(x)} \cdot f'(x) = \frac{x}{y} \frac{dy}{dx}, \quad (1.3)$$

provided that the limit exists. For a fixed  $x = x_0$  the number  $E_f(x_0)$  is said to be **the elasticity of  $f(x)$  at  $x_0$** .

With a view to (1.2) the elasticity of the function  $y = f(x)$  is **the instantaneous (rate of) proportional change in  $y$  with respect to proportional change in  $x$** , or equivalently it gives **the approximate proportional change in  $y$  as a response to a unit proportional change in  $x$** . Since

$$\frac{d(\log y)}{dx} = \frac{1}{y} f'(x) = \frac{f'(x)}{f(x)},$$

the formula (1.3) may be expressed in the form

$$E_f(x) = \frac{d(\log y)}{d(\log x)} = \frac{x}{y} \frac{dy}{dx}. \quad (1.4)$$

The important property of the elasticity is that it is a number independent of the units in which the variables are measured. It is a consequence of the fact, that the elasticity is defined in terms of proportional changes. It may be simply verified—if  $x' = rx$ ,  $y' = sy$  are the new measures of  $x$  and  $y$  respectively, then

$$\frac{x' dy'}{y' dx'} = \frac{rx d(sy)}{sy d(rx)} = \frac{rx sdy}{sy rdx} = \frac{x dy}{y dx},$$

i.e. the elasticity (see (1.4)) is unchanged.

There is one more useful (sometimes initiating the elasticity) verbal transcription of the elasticity, namely in terms of percentages. It holds

$$\Delta x\% = \frac{\Delta x}{x} \cdot 100,$$

and due to (1.2) we get

$$\frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y\%}{\Delta x\%}$$

and hence from (1.3) it follows, that the elasticity  $E_f(x)$  measures the instantaneous rate of percent change in  $y$  with respect to percent change in  $x$ , or equivalently it shows the approximate percent change in  $y$  as a response to a unit percent change in  $x$ .

Since the elasticity is based on the derivative, the basic rules for the evaluation of elasticities may be simply derived (see Allen (1968) among others). We will not need them in the sequel. Notice that unlike the derivative, the linear function  $y = ax + b$  has not constant elasticity, its elasticity is  $\frac{ax}{ax+b}$ . On the other hand, the power function  $y = ax^\alpha$  ( $\alpha > 0$ ) has constant elasticity  $\alpha$ .

## 2 Elasticity in economics

The concept of the elasticity plays a crucial role in analyses and decision making in economics. We will present some straightforward and convincing examples.

### The elasticity of demand

The market demand for a product depends on many factors—the price, the average income, the prices of alternative goods, the expectation, the taste etc. In a simplified, so called *one-factor model*, suppose that *the quantity purchased*  $Q$  depends only on *the price*  $P$ . Changes in the price  $P$  will lead to changes in the quantity purchased  $Q$ . This may be represented, under certain “normal conditions” by the decreasing function  $Q = D(P)$  (or its inverse) which is referred to as *a demand function*. Its elasticity will measure the response (or the sensitivity) in  $Q$  to change in  $P$ . Applying (1.3) we get (inserting the sign minus to reach the positivity of the result)

$$E_D = E_D(P) = -\frac{P}{Q} \frac{dQ}{dP} = -\frac{P}{Q} Q', \quad (2.1)$$

where  $Q = D(P)$ . Note (as given above), that  $E_D(P)$  is independent of both price and quantity units. Now we will demonstrate, how the elasticity directly affects decision making in economics. The seller may change the price to influence the quantity purchased  $Q$  in order to maximize *total revenue*  $TR$ , which is given by

$$TR = P \cdot Q = TR(Q), \quad (2.2)$$

where  $P = D(Q)$  is a demand function (in inverse form). Now, some mathematics is needed to get criteria for decision making. We will examine the derivative  $TR'$  of  $TR$ , which is referred to as the *marginal revenue*  $MR$ , i.e.

$$MR = MR(Q) = TR' = \frac{dTR}{dQ}.$$

It holds (see 2.2)

$$\begin{aligned} MR = TR' &= (D(Q) \cdot Q)' = Q' \cdot D(Q) + Q \cdot D'(Q) = D(Q) + Q \cdot D'(Q) = \\ &= P + Q \cdot \frac{dP}{dQ} = P \cdot \left(1 + \frac{Q}{P} \frac{dP}{dQ}\right) = P \cdot \left(1 - \frac{1}{\frac{P}{Q} \cdot \frac{dQ}{dP}}\right). \end{aligned}$$

Using (2.1) we obtain

$$MR = P \cdot \left(1 - \frac{1}{E_D(P)}\right). \quad (2.3)$$

Now, we will analyze  $MR$  with respect to the values of  $E_D(P)$  (viewing (2.3)):

<sup>1</sup> If  $E_D(P) > 1$ , then  $MR > 0$  and hence  $TR$  increases, i.e. if  $Q$  increases, then  $TR$  increases. But the increase in  $Q$  is connected with the decrease in  $P$ . From it we deduce:

**Conclusion 1.** If  $E_D(P) > 1$ , then a small decrease in price results in an increase in total revenue and a small increase in price results in a decrease in total revenue. This case is in economy referred to as **elastic demand**.

Then the following decision procedure is justified :

*Initial situation:*  $E_D(P_0) > 1$  for some price  $P_0$ , i.e. demand is elastic at  $P_0$

*Task:* to increase total revenue  $TR$

*Economic decision:* reasonably small decrease in price.

<sup>2</sup> If  $E_D(P) < 1$ , then  $MR < 0$  and hence  $TR$  decreases, i.e. if  $Q$  increases, then  $TR$  decreases and consequently if  $Q$  decreases, then  $TR$  increases. Analogous to the above case the decrease in  $Q$  is connected with the increase in  $P$ . From it we deduce:

**Conclusion 2.** If  $E_D(P) < 1$ , then a small increase in price results in an increase in total revenue and a small decrease in price results in a decrease in total revenue. This case is in economy referred to as **inelastic demand**.

Then the following decision procedure is justified:

*Initial situation:*  $E_D(P_0) < 1$  for some price  $P_0$ , i.e. demand is inelastic at  $P_0$

*Task:* to increase total revenue  $TR$

*Economic decision:* reasonably small increase in price.

The remaining special case  $E_D(P) = 1$  signals a maximum value of total revenue. There is one practical question concerning elasticity of demand. If the demand function is known from empirical investigations, then not much complicated calculation



is needed because the approximation functions are mostly of polynomial type. But the bulk of initial situations to start decision procedure is based on intuitive engineering reasoning. To state whether at a given price (or range of price) the demand is elastic or inelastic, economic experts rely on their experience and the character of a commodity with a view to its sensitivity to price changes. While for instance the demand for medical services, public transport, basic foods is typically inelastic, the demand for luxury goods or goods with many substitutes is typically elastic. To make a qualified economic decision the knowledge of the above mentioned theoretical arguments is claimed.

### The elasticity of total cost

Extremely useful application of the elasticity concept is in the analysis of the cost problem. A **total cost function** is of the form  $TC = TC(Q)$ , where  $Q$  is an output produced by the firm. The elasticity of total cost is given (applying (1.4)) by

$$E_{TC}(Q) = \frac{Q}{TC} \cdot \frac{dTC}{dQ}, \quad (2.4)$$

where  $\frac{dTC}{dQ}$  is referred to as a **marginal cost**, denoted by  $MC$ , and  $\frac{TC}{Q}$  an **average cost**, denoted by  $AC$ . Obviously it holds

$$E_{TC}(Q) = \frac{MC}{AC}. \quad (2.5)$$

From (2.4), (2.5) it follows:

1<sup>0</sup> If  $E_{TC}(Q) < 1$ , then a small proportional increase in output  $Q$  is obtained at a less than proportional increase in total cost, average cost is greater than marginal cost and average cost decreases as output increases. This case is in economy referred to as **increasing returns**.

2<sup>0</sup> If  $E_{TC}(Q) > 1$ , then the situation is exactly the reverse of that when  $E_{TC}(Q) < 1$ . This case is in economy referred to as **decreasing returns**.

Now, for the cost analysis, economists state two basic the normal case of cost conditions (A), (B) (Allen (1968)):

- (A) Total cost increases continuously from positive value (fixed cost) as output increases from zero.
- (B) The elasticity of total cost increases continuously from values less than 1 at small outputs to values greater than 1 at large outputs. In other words, returns become increasingly economically unfavourable as output increases.

From the above assumptions and conditions (A), (B) it follows, that there exists an output  $Q = Q_0$  such that  $E_{TC}(Q) = 1$  and returns cease to be increasing and become decreasing. Further, average cost  $AC$  decreases with increasing output at first,

reaches a minimum value at  $Q_0$  and then increases as output increases. Marginal cost  $MC$  is less than average cost  $AC$  for outputs less than  $Q_0$  and greater than average cost  $AC$  for outputs greater than  $Q_0$ . Now, using standard calculus tools, simple mathematical models of the total cost function satisfying the normal case of cost conditions may be derived. The most convenient function is the cubic polynomial

$$TC = TC(Q) = aQ^3 - bQ^2 + cQ + d,$$

where  $a, b, c, d$  are positive constants with  $b^2 < 3ac$ .

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# A Sense of Proportions in Engineering Mathematics

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Mathematics courses for engineering students in Karlstad traditionally start by repeating what the students are supposed to have studied at school. For the well prepared student this is a waste of time.

During spring 2004, I have tried a different introduction, starting with

- (i) topics, new to the students,
- (ii) familiar topics, but viewed from a new perspective.

The new topics were from linear algebra; vectors, lines and planes in three dimensions, matrices, and systems of linear equations. Here we begin from a basic level and, since these are new topics, they require no repetition of material previously studied by the students. I shall say nothing further about these topics here.

Instead, I will focus upon alternative ways of viewing familiar concepts. The issues I intend to discuss are fractions, proportionality, and exponential growth.

## 1 Background

In [1], Wallin quotes de Bock et. al. [2] who, in their turn, quote Freudenthal [3]:

The insight that volume is proportional to the cube of length, and that area is proportional to the square of length, is one of the most fundamental insights of all mathematics.

(The quote is not verbatim as I have been unable to find the exact phrasing.)

[2] describes a study of 120 pupils in grade seven (around 14 years of age) given two types of problems. Examples of the different types were (free translation):

- A.** Farmer *A* needs 4 days to dig a ditch round his square field with side 100 m. How long would it take him to dig a ditch around another square field, with side 300 m?

- B.** Farmer  $B$  needs 8 hours to spread fertilizer over his square field with side 200 m. How long would it take him to spread it over another square field, with side 600 m?

It appeared that type **A** was correctly solved by around 90–95% of the pupils and type **B** by around 5% of the pupils. The result is interesting, if not very surprising. We might expect better from students beginning their engineering studies. However, I frequently encounter errors from engineering students, based on the misconception that “everything is linear”. When designing the new course, it seemed reasonable to confront the students with various examples of non-linear models.

## 2 Highlighted topics

### 2.1 The first problem

Example 1 below was given to the students as a task to be solved at home:

**Ex. 1.** 15 men can build a wall 33 m long,  $1\frac{1}{4}$  m wide and  $3\frac{1}{2}$  m high, in three weeks if they work 9 hours a day. What is the time required for 18 men to build a wall 27 m long,  $1\frac{2}{3}$  m wide and  $2\frac{4}{9}$  m high, if they work 10 hours a day?

I discovered that most students started calculating “the time required for one man to build 1 cubic metre of wall”. I presented another solution, based on the idea that the time needed, say  $t$ , would be proportional to some of the quantities, and inversely proportional to some of the others. This led to the following equation

$$\frac{t}{3} = \frac{15 \cdot 27 \cdot 1\frac{2}{3} \cdot 2\frac{4}{9} \cdot 9}{18 \cdot 33 \cdot 1\frac{1}{4} \cdot 3\frac{1}{2} \cdot 10}. \quad \text{Cancelling gives } t = \frac{12}{7} \text{ weeks (12 days?).}$$

Obviously, one might discuss whether the model is realistic. Be that as it may, I find this a useful way of thinking, and presented some material to develop the theme.

### 2.2 Fractions

Given a fraction with common factors in numerator and denominator most students are likely to start looking for the common factors to be cancelled by division, e.g.

$$\frac{21}{28} = \frac{21/7}{28/7} = \frac{3}{4}.$$

Instead, I have suggested factorization:

$$\frac{21}{28} = \frac{3 \cdot 7}{4 \cdot 7} = \frac{3}{4}.$$

The point is that one need not get stuck if the common factor is hard to see, e.g.

$$\frac{74}{111} = \frac{2 \cdot 37}{3 \cdot 37}.$$

It is also simpler for cases when you are multiplying fractions, like

$$\frac{48}{49} \cdot \frac{63}{64} \cdot \frac{80}{81} = \frac{6 \cdot 8 \cdot 7 \cdot 9 \cdot 8 \cdot 10}{7 \cdot 7 \cdot 8 \cdot 8 \cdot 9 \cdot 9} = \dots,$$

and it would certainly be preferable in most algebraic situations, like

$$\frac{x^2 - 1}{x^2 - x} = \frac{(x + 1)(x - 1)}{x(x - 1)} = \frac{x + 1}{x} \quad (\text{assuming } x \neq 0, 1).$$

### 2.3 Proportionality

Consider the following problem:

**Ex. 2.** 210 g of cheese cost 12 shilling. What is the price for 560 g of the same cheese?

*Comment:* In my experience, most students start calculating the price for 1 kg and multiply by 0.56. However, this is a detour, as we are not required to compute the constant price per unit weight. Moreover, this procedure would include a rounding-off error.

Instead, the model “price over price equals weight over weight” is advocated, i.e.

$$\begin{aligned} \frac{p}{12} &= \frac{560}{210}, \\ p &= \frac{12 \cdot 10 \cdot 7 \cdot 8}{10 \cdot 3 \cdot 7} \quad \text{etc.} \end{aligned}$$

The issue about an un-called-for constant is emphasized in the next example:

**Ex. 3.** In certain conditions, when the brakes are applied to travelling a vehicle at 40 km/h, it will come to a halt after 48 meters. What is the distance required if the vehicle is travelling at 30 km/h when the brakes are applied? (Assume constant acceleration.)

*Comment:* We might compute the acceleration ( $\approx -1.286$  m/s), but this is a tedious solution, involving different units for time and other complications. If instead, we use the fact, imposed on us when we study for a driving license, that the distance in question is proportional to the square of the speed, we get:

$$\frac{s}{48} = \left(\frac{30}{40}\right)^2,$$

which is easier to solve.

Note that the answer remains the same if we change 40 and 30 km/h to 4 and 3 speed units respectively, whereas the acceleration constant will change.

**Ex. 4.** An atlas covering Ruritania at a scale 1 : 300 000 has 36 pages. How many pages would be required for a Ruritanian atlas of equal dimensions at a scale 1 : 400 000?

*Comment:* To determine a constant of some sort here turns out to be awkward, as neither shape or real area of the country nor the length or width of the atlas are known, so the simplest solution is to use that area scale is proportional to squared length scale.

A similar procedure is useful in the next example:

**Ex. 5.** Consider the function

$$f(t) = \frac{c\psi \ln t}{\lambda e^{\varrho+\psi} \sqrt{\pi}},$$

where  $\psi$  is a constant,  $c = 1.27$ ,  $\lambda = 0.823$  and  $\varrho = 6.31 \cdot 10^{-7}$ . Experiments have shown that  $f(2048) = 1.65$ . Determine  $f(1024)$ .

*Comment:* Most students, equipped with calculators, seem to find this task pretty challenging, as the constant  $\psi$  has two possible values, none of which is trivial to find. With or without technical aid, the idea that

$$\frac{f(1024)}{f(2048)} = \frac{\frac{c\psi \ln 1024}{\lambda e^{\varrho+\psi} \sqrt{\pi}}}{\frac{c\psi \ln 2048}{\lambda e^{\varrho+\psi} \sqrt{\pi}}} \quad \left( \text{cancelling into } f(1024) = 1.65 \cdot \frac{10}{11} \right).$$

seems a surprisingly new one to the students.

## 2.4 Exponential growth

The general procedure in cases of exponential growth is to use the formula

$$y(t) = y(0) e^{ct}, \quad (1)$$

or alternatively,

$$y(t) = y(0) 10^{kt}. \quad (1')$$

But, if we know, say, that a population increases by 70% in thirteen years, it is easier to define a time-unit of 13 years, and write

$$\frac{y(t)}{y(0)} = 1.7^x, \quad (2)$$

where  $x$  is the number of thirteen-year units.

**Ex. 6.** From a well known Swedish text-book in mathematics I quote:

*In the spring 1937, 8 pheasants were left on a desolated island. ... In the spring 1941 the number of pheasants was 705. How many could be expected in spring 1942 [assuming exponential growth]?*

The book uses the equation  $y = 8 \cdot 10^{kx}$  ( $y$  number of pheasants  $x$  years after 1937) to get

$$k = \frac{\lg\left(\frac{705}{8}\right)}{4} \approx 0.486,$$

which gives the answer  $8 \cdot 10^{5-0.486} \approx 2153$  pheasants. This problem appears in several editions of the book, with the same solution.

*Comment:* A simpler way would be to define four years as 1 time unit. Then the rate of change per time unit is  $\frac{705}{8}$ . 5 years equal  $\frac{5}{4}$  time units, thus  $\frac{y}{8} = \left(\frac{705}{8}\right)^{\frac{5}{4}}$  gives  $y \approx 2160$ ; a quicker, simpler and more accurate result, as we avoid intermediate round-off errors. (The accuracy point turns out to be less important as, due to food shortage, the true value was 1325.)

**Remark.** In example 6, there is a risk that obstinate students stick to the familiar procedure based upon equation (1) or (1'). This would spoil the attempt to teach a new way of viewing the situation.

One strategy for imposing the “new” way of modelling the problems is to give numerical values which are convenient for the proposed model and to demand that the problems be solved without calculator. A serious draw-back with calculators in situations of mathematics learning is that the calculator is so powerful that even very clumsy solutions may be quicker than the time needed to grasp the elegant solution. This hampers the development of cognitive tools.

**Ex. 7.** A radioactive substance is reduced by 36.0% in 28 minutes and 16 seconds.

- a) Without calculator, compute the reduction in half the time.
- b) How long does it take before 48.8% of the substance has vanished.

*Comment:* In 7a) the remaining factor is 0.64, thus a convenient square, so in half the time the substance is reduced by 20%.

In 7b) the remaining factor is 0.512, thus  $0.64^{3/2}$ , so the time required would be a one and a half 28-min-16-sec-time-unit, or 42 minutes and 24 seconds.

### 3 A simple study

At the end of my course the students had a written exam. One problem was the following:

**Ex. 8.** The diagonal of the sitting-room in a flat is 9 m. On a map of the flat the same diagonal is 45 mm. The total area of the flat is 22 square cm on the map. What is the real area of the flat?

The result was depressing. Little more than half of the students managed to solve the task correctly. Among the errors were the belief that the area computed was the area of the sitting-room, not of the entire flat, and in that case there is no solution. There were also several attempts with a linear model, which gives the area  $4400 \text{ cm}^2$ . Some students realized that this would call for a very narrow flat, but others changed the units to  $44 \text{ m}^2$ . This confirms the persistence of deeply rooted misconceptions.

## 4 Concluding remarks

The ideas presented above are most definitely not new. But my impression is that they have vanished from general use in mathematics teaching during the recent decades, presumably in the wake of calculators invading maths' class-rooms.

I believe many university mathematics teachers have occasionally wondered how the students have spent their pre-university careers; it often happens that one must repeat even very fundamental procedures from elementary levels.

The danger is that repetition may be boring to those students who do not need it. But, in real life, it may be of extreme importance that a result is correct. For an engineer, the ability to find different ways of checking a crucial calculation is a fundamental one. Moreover, in suggesting alternative ways of solving old problems, we can challenge the whole group of students.

The impact on the weaker students was a disappointment. My impression, however, is that the better students did learn something essential from the exercises.

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# The Future of Mathematics in the United Kingdom

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## Introduction

The decline of mathematical ability of undergraduate entrants to undergraduate engineering courses in the United Kingdom has been well documented. The publication this year of a report into mathematics education post-14 makes some recommendations for rescuing the situation.

## 1 What is the mathematics problem?

Three aspects of mathematics education in the United Kingdom are of particular concern: the number of candidates for A-level mathematics, the lack of basic mathematical knowledge and skills among university entrants and the shortage of mathematics teachers. These are examined, in turn, in this Section.

### 1.1 Candidate numbers

The number of A level candidates in 1991 was 74 972; this had fallen to 64 605 by 1995, including a drop of over 6 000 between 1992 and 1993. Small increases in the next four years led to false hopes that the decline had been arrested, but the decline started again in 1998. By 2001 the number had dipped to 65 891 and then suffered an 18% fall to 53 940 in 2002, followed by a slight increase to 55917 last year. The fact that there was a failure rate of some 29% in AS mathematics the previous year was clearly the over-riding factor. In 2002 the failure rate in AS mathematics had reduced but remained substantially above 20%, significantly higher than in other subjects. The revisions that are being introduced will not have an effect until 2005.

### 1.2 Knowledge shortfall

There is well-documented concern that students arriving at university now show less mathematical fluency than their counterparts of a decade ago; see, for example, Mustoe (2002), Mustoe and Lawson (2002) and the reports Engineering Mathematics Matters (1999) and Measuring the Mathematics Problem (2000). What we are discussing is really basic mathematics - simple algebraic manipulation, simple ideas

in calculus etc; in fact, the sort of mathematics of which we could assume ten years ago that the new students would have a firm grasp. These lacunae are not easily or speedily filled. A major problem has, and remains, the gap between GCSE mathematics and its A level counterpart. The fact that candidates can gain a grade C on little more than 22% of the raw marks and a Grade B by taking the two middle papers, and hence having little exposure to algebra is deeply worrying.

### **1.3 Teacher shortages**

Despite the efforts of the Teacher Training Agency, the recruitment to target of mathematics teacher trainees has failed to reach 80% of the places made available. Last year, the initiatives aimed at increasing recruitment resulted in the disappointing figure of 78 extra teachers successfully completing the course. To meet current intake targets about 38% of graduates with a first degree in mathematics would be required to embark on training; this will not happen. Added to that is the fact that a sizeable proportion of the total mathematics teaching force is over 50 and the future does not seem rosy.

### **1.4 Will the situation improve?**

It would take an extreme optimist to believe that things will not get worse before they start to get any better. Any noticeable improvements will take some time to materialise. Is it being pessimistic to forecast that such improvements will not occur in the foreseeable future?

## **2 Making Mathematics Count—the Report of the Inquiry into Post-14 Mathematics Education**

On February 24<sup>th</sup> this year the long-awaited report of the Government inquiry, led by Professor Smith, into mathematics education post-14 was published.

In his foreword to the Secretary of State for Education and Skills, Professor Smith highlighted the deep concerns expressed by so many “important stakeholders” about the learning and teaching of mathematics in England: there was a widespread belief that the situation had reached crisis level. The Report identifies three key areas of especial concern: there was a critical shortage of specialist mathematics teachers; the current framework of the curriculum and qualifications failed to meet the needs of “learners, higher education and employers”; there should be provision for supporting those currently teaching mathematics via continuing professional development and other resources.

### **2.1 Outline of the Report**

The Inquiry got under way in late 2002 in response to growing disquiet being expressed about mathematics education in the U.K. The disastrous failure rate at AS mathematics in 2001 and the subsequent drop of almost 20% in A level entries for

the following academic year (on top of a fall of almost 10% in entries during the previous decade) finally seemed to have made the alarm bells audible. The continued failure to recruit and retain specialist teachers of mathematics, despite the best efforts of the Teacher Training Agency, had put the U.K. in a situation where well over one-third of each year's graduates in mathematics would need to go into teaching for each of the next few years just to close the gap, and that takes no account of the age profile of teachers in post, which indicates a much higher number than currently who will be reaching retirement age in the near future.

The report emphasises the importance of mathematics in today's economy and highlights the breadth of career opportunities for mathematics graduates; in a sense, mathematics has been a victim of its own success - teaching mathematics is perceived by many graduates as low down on the scale of attractive careers and some clear incentives are needed to attract graduates into the teaching profession in sufficient numbers.

Having put the case for the importance of mathematics the report takes a detailed look at the supply of teachers of mathematics, reviews current pathways in mathematics education and lays out suggested actions on these pathways and possible future pathways. Support for the teaching and learning of mathematics is considered and the need for national and regional infrastructures is argued.

## 2.2 The recommendations

In all, the report lists 44 recommendations, some of which can and should be implemented fairly swiftly, whilst others require much resourcing and considerable structural change and will require a longer time-scale to bring into play.

There should be general agreement in the U.K. mathematics community on many of the recommendations. However, one recommendation that has already proved controversial concerns the role of Statistics and Data Handling in the GCSE (16+) mathematics syllabus. Professor Smith, himself a statistician, suggested that much of the topic could better be taught in an integrated manner in other subjects which make use of the techniques, thereby freeing up time in the mathematics timetable to the acquisition of mastery of "core mathematical concepts".

The first two recommendations are designed to give mathematics a higher profile. It is suggested that a high-level post within the Education department needs to be created for someone to have a specific responsibility for the subject. The Advisory Council on Mathematics Education should have increased support and a similar body should be established to carry out a parallel role with regard to strategic issues in research and knowledge transfer.

Recognising the need for fresh incentives to attract more graduates into teaching it is suggested that the possibility of enhanced remuneration for teachers of subjects where there is a shortage, such as mathematics, should be re-examined. In addition, fast-tracking towards teacher certification could provide an additional supply of teachers, albeit up to Key Stage 3 only, for example.

The seeming failure of GCSE to meet the needs of its constituents led the Inquiry to recommend a two-tier system for GCSE mathematics and to make the subject a double-award one like science, in recognition of the amount of work it requires for

success. One bone of contention has been that GCSE mathematics does not stretch the most able, and the Inquiry asks for special attention to be paid to this aspect. One size clearly does not fit all, no matter how it is packaged. The catastrophe that was AS mathematics in 2001 has resulted in attempts to ameliorate the situation by reducing syllabus content. If the next few years do not see a significant improvement in numbers taking mathematics post-16 the Inquiry suggests that radical measures, including some form of financial inducement, be considered—another recommendation which is likely to attract considerable opposition.

A strong element of the recommendations is the provision of fully-resourced support for mathematics teachers in the form of CPD which might be rewarded financially. Laudable as this idea is, it is going to take a strong shove from Government and a culture shift to get it implemented. Professor Smith states that about a quarter of mathematics teachers currently employed spend part of their time not teaching mathematics, and when you ask who is going to cover for mathematics teachers whilst they undertake their CPD the scale of the task comes sharply into focus.

As a mechanism for the provision of this support it is proposed that a National Centre for Excellence in Mathematics Teaching be established, together with nine Regional Mathematics Centres. In addition to supporting the delivery of CPD, the infrastructure should provide both a strategic co-ordination of and local support for a wide range of resource provision for the support of the teaching and learning of mathematics. Among elements to be considered are a resource for dissemination of educational research (including those relating to the use of ICT); networking with local schools, colleges, higher education and business and building on relevant existing mathematics support activities and initiatives.

In summary, the Inquiry has identified three areas of especial concern: the shortage of specialist mathematics teachers in schools; the failure of the current curriculum and qualifications framework to achieve fitness for purpose; the need to support current teachers of mathematics through CPD *inter alia*.

### 3 Conclusions

The message from the Inquiry Report is clear: unless the U.K. Government acts swiftly, decisively and fearlessly then mathematics education is in danger of terminal decline. That the response has, so far, been underwhelming is a matter for deep concern. “Let the shipwrecks of others be your sea-marks” (attributed to Tony Hancock).

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# Math on the Web: Do Students Prefer Computers and E-learning or do They Stick to Their Teachers?

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E-learning is a rapidly growing activity, and more and more students get their education through the web-based courses. It seems there is a competition going on, among different institutions, in offering web-based education. Until now, we have seen only very few reports about the students' opinions of such education.

Mathematics has a negative development at universities and colleges. Numbers of students in mathematics courses and those passing the exams successfully are rapidly decreasing. To overturn this negative development, it may be necessary to teach in a different way, where new technology could play an important role. The Xmath project, which was a project within the EU Socrates Minerva programme scheme, was an attempt to meet these challenges. The aim of the project was to use new technology to present mathematics on the web, and to make a Pilot course for engineering students and for teacher training students available free on the WWW.

The pilot course consists of theory, applications, problems and tests, in addition to a web-based step-by-step calculator and a chat programme made for scientific talk.

The Pilot course has been tested by teacher training students in Norway and engineering students in Slovakia and in Spain. Students were asked to give comments on the contents, structure, educational calculator and chat programme. They were also asked to give comments on their own opinion and experience with the e-learning in general.

## **Evaluation of the pilot course in teacher training education in Norway**

The pilot course was tested by a group of students following a calculus course in the academic year 2002/2003. The students were partly off-campus. There were 17 students following the course. 7 of them were students in the teacher training education. The others were external students, trained teachers who were partly students and partly working.

The students had 5 lectures during the term, and because of the few lessons the students got, they had to study a lot on their own, and they could not be taught all the content they were expected to learn. The lessons were focused on the harder concepts. Other concepts were discussed as they came up naturally in these lessons.

The students had a Norwegian textbook. They were organised into groups of 2–5 students, but two students were on their own. Some of them had physical meetings, others communicated by mail, phone and a learning management system.

On these conditions the pilot course had to be a supplement for the students. They were given a group paper connected to the course where they were told to evaluate both the contents of the course, and how it functioned pedagogically. They were asked to use the web-calculator, and give an evaluation of it. They were told to try Scientific Talk, give comments on its functionality and suggest improvements.

## **Students' opinions**

### **Contents:**

- the content is relevant according to the aims and objectives of their course
- the pilot course can work as a supplement to their text book

### **Composition and representation:**

#### *Positive*

- The structure of the main page
- The size of the text, which makes it easy readable
- Illustrative figures placed inside the text

*Negative*

- Too much scrolling
- The use of colours are not consequent
- Blue colours are associated with hyperlinks

### **3.1 Representation of the contents:**

*Positive*

- The course work as a place to find answers to the questions they might have
- The groups are positive to the pedagogical way of presenting the content
- They are also positive to the use of tests, problems and examples

*Negative*

- One group miss a search function
- One group would like a better system of where to find problems and tests according to different subjects
- All groups mention in different places in their report that they have problems with the language, and would like to have a Norwegian version

### **The different modules:**

- The students want all modules to have the same structure
- The use of frames are discussed, some students like to have a “working window”, others do not

### **Xmath Educational Calculator:**

- The students find it easy to use, except for writing the mathematical expressions
- They are very satisfied with the step by step function.

### **Scientific Talk:**

- None of the students are familiar with LaTeX, and find the use of symbols panels very time-consuming
- They want “pull down” menus
- Writing text is no problem.



**General:**

- Many of these students have a job in addition to their studies. They prefer net based education, since they do not have to leave their job and family.
- The students find it very time consuming to write mathematics that can be sent via Internet
- They are concerned about the communication with other students and teachers
- They find it important to have real meetings and software that makes communication and collaboration easily.

**Evaluation of the pilot course in engineers in Spain**

The pilot course was tested by a group of 50 students following a standard calculus course in the academic year 2003/2004. The learning used is classical learning: Lectures, practical lessons and laboratories.

The target group consisted of only 20 students from the group. Students had 2 lectures per week and 3 practical lessons, some in the maths lab. They used a Spanish textbook written by several Spanish teachers.

Students were free to visit the Pilot course at their convenience. Some times the professor advised it could be better to run some modules with some colleague to comment, discuss, etc.

On these conditions the main goal of the Pilot course was to enhance the mathematical concepts involving the topics which the Pilot course is dealing with.

**Some restrictions:**

We did not use other facilities of blackboard like Scientific Talk because the students could join the teacher every day.

It was not possible to use the Xmath Education Calculator as it was not working.

Questions submitted to students were the same as used in the Questionnaire for Norwegian students. We have added one more question about the time used for running the Pilot course.

**Students' opinions****Contents:**

- Sometimes the concepts are introduced with lower deep we need. In other topics we can refresh concepts of Secondary School.
- In general the pilot course can work as a supplement to textbook.

## **Composition and representation:**

### *Positive*

- The structure of the main page.
- The size of the text, which makes it easy readable
- Illustrative figures placed inside the text
- Links to other pages

### *Negative*

- Lack of homogeneity between the pages

## **Representation of the contents:**

### *Positive*

- The students are satisfied with the course. After lectures they are glad to run the course. They can understand all topics we are teaching in the lectures more easy.
- The students are positive to the pedagogical way of presenting the content, and also the use of problems, tests, puzzles, exercises, etc.

### *Negative*

- Language—preferably Spanish

## **The different modules:**

- Students want all modules to have the same structure—very important
- More easy navigation in Differentiation and Integration modules

## **Xmath Educational Calculator:**

- Not working

## **Scientific Talk:**

- Not used

**General:**

- Students agree with the goal: to enhance the traditional method of teaching.
- They liked to deep in distance learning for traditional Universities: tutorials by e-mail, files sent to professor, etc.
- The professor is important.
- Students had not much time to run the course. They have a strict curriculum and they need to study a lot of topics. The average time of running the course was 12 hours per student.
- For our students this course could be an introductory course in Calculus I. We need more topics and more depth in the topics.
- More tests and exercises will be welcome.

**Evaluation of the pilot course in engineering education in Slovakia**

The pilot course was tested by 3 different groups of students. There were 13 Slovak students and 2 English speaking foreign students in the first year of their study at the Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, taking the basic course on Mathematics I in the English-speaking group. 3 responses to the questionnaire we have received from the English speaking students in Australia, who were not studying Mathematics, but wanted to be volunteers to go through the text and to check their owns mathematical abilities and understanding.

In the basic course Mathematics I students had 2 hours of lectures and 2 hours of practical lessons per week, every second week in the computer lab. They used a Slovak textbook written by their teachers. Students were encouraged to use the Pilot course on their own, as an additional source of study material available in the computer lab. They used it not during the practicals, but in their free time.

Some students took their task seriously and tried to help in order to create a good source of teaching material. Some were very sceptic and did not regard electronic sources of teaching materials as very useful because of the lack of access to computers and Internet at our university.

All students but the 2 from Australia considered the direct student—teacher communication as a crucial way of delivering knowledge in the learning process. They would hardly accept the Pilot course as the only one source material of explaining the mathematical concepts in a comprehensive way.

## Students' opinions

### Contents:

- Some students found the text very instructive, useful and related to their course.
- They used it as a supplement to their textbook in Slovak, mostly to find English terms and expressions. For learning they used primary their Slovak textbooks, which was easier for them to understand concepts.
- Some students complained about difficult text to understand (especially in Numbers and Functions) with too many new concepts used in explanations and descriptions. They found the text to be a not sufficient study material.
- Students pointed to many grammar mistakes and not finished ideas and complicated sentences.
- Some concepts were explained by other unknown concepts, so the whole text was very difficult to read and understand.
- They did not think it would be a sufficient material on its own.

### Composition and representation:

#### *Positive*

- Well arranged individual sections
- Good structure and layout of the main page
- Readable text in good size, font and different colours
- Lot of illustrative figures in the text
- Many links to other pages

#### *Negative*

- Some unknown and strange symbols
- Different navigation in different modules
- Not all links were working
- No homogeneity between the different modules

**Representation of the contents:***Positive*

- An easy way to get an answer to some problems with basic terms and notions
- Each student can work at his/hers own place: at home, in Internet cafe, in the school computer lab, wherever
- Easy access to other sources of information if necessary (the links to other sites)
- Lot of additional explanatory material—illustrations, applications, stories, puzzles, exercises, solved problems, etc.

*Negative*

- Not sufficient text for finding answers to questions regarding understanding
- The shift in writing style—sometimes the text is very clear, using basic language but at times the language becomes too obscure with too many technical terms that confuse rather than clarify the concepts
- Slovak students would prefer Slovak language

**The different modules:**

- Majority of students liked more the colourful and structured design of modules Calculus—Differentiation and Integration.
- They did not mind the different design; they pointed to problems with getting used to it.
- Some students found modules Numbers and Functions as easier, because there were no frames—windows to follow simultaneously on the screen.
- Most students preferred the page layout of Calculus modules; they liked the idea to have the text in one part, and examples and other supplementary material in the other part.
- It is clearly marked what type of information (input) is being received, thus the student does not get confused.
- The theory is kept separate from additional information, thus making a clear distinction between information.
- The text is clear and easy to read in Calculus modules, more difficult and incomprehensible in Numbers and Functions.

**Xmath Educational Calculator:**

- All students responded that the calculator was not working at all.

### **Scientific Talk:**

- Students did not use the programme; they did not feel like to do so and for what reasons. (In fact, they were not obliged to do so.) They were not sure how to use the programme. Perhaps a brief explanation on usage once logged on would be useful.

### **General:**

- Students mostly did not like the idea of a net-based course without real classes (12).
- They were not sure they could understand the mathematics well without face-to-face explanation and discussion with the teacher.
- Electronic courses on Internet can be very useful, particularly the easy access to any resources, to discussion boards, etc.
- Explanations need to be very clear and simple. Without the 'human touch', students can often get confused; therefore such programmes should aim to be clear and easy to use.
- Students were not sure if they would prefer to apply for an Internet e-learning course on Mathematics, probably yes, but only for enhancing the traditional courses.

# New Initiatives in Teaching Mathematics to Students of Sports Technology

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## Abstract

A new mathematics module, for Sports Technology students, was introduced at Loughborough University for the academic year 2003–4. This paper gives reasons for, and describes, the new initiatives which feature in this module. These include small group teaching, the introduction of a computer algebra system, Matlab, the introduction of group projects into the assessment process and finally the inclusion of sports related problems into the syllabus. It then reports on the success of the initiatives. Attendance by the students on the new module and their end of year results are compared with the previous year's figures. Motivation of the students is also discussed. Finally the paper provides comments upon some of the issues to be addressed if others wish to adopt some or all of the features introduced in this module.

## Introduction

Loughborough University is renowned throughout the UK as being one of the leading universities for sport. Top sports scholars are attracted to Loughborough to study and train. Many wish to study sports related subjects and in recent years a course in Sports Technology has been developed. This course focuses primarily upon the design and manufacture of sports equipment. It is supported with a background of manufacturing technology, engineering science, mathematics, statistics and experimental design.

The Mathematics Education Centre (Croft and Robinson, 2003), is responsible for the teaching of mathematics to the majority of engineering students at the university and teaches mathematics to the Sports Technology students. Prior to 2003–4 these students were taught mathematics alongside Manufacturing Engineering students in a class of approximately 100 students. However there was poor attendance by Sports Technology students at mathematics tutorials and a high failure rate in mathematics. The reasons were varied. Some did not see the relevance of mathematics to their course and therefore were not motivated to study the mathematics module. Others found the transition from school to university mathematics difficult.

It was decided that a new mathematics module, for first year Sports Technology students only, would be introduced for the academic year 2003–4. This coincided

with a lowering of the mathematics entry requirements for the course. The author was assigned to teach the new module.

This paper gives reasons for, and describes, the new initiatives which feature in this module. These include small group teaching, the introduction of a computer algebra system, Matlab, the introduction of group projects into the assessment process and finally the inclusion of sports related problems into the syllabus. We then report on the success of the initiatives. Attendance by the students on the new module and their end of year results are compared with the previous year's figures. Motivation of the students is also discussed. Finally the paper provides comments upon some of the issues to be addressed if others wish to adopt some or all of the features introduced in this course.

## **The New Initiatives**

Four major changes were introduced in the mathematics module for first year sports technology students in the year 2003–4. The reasons for introducing these initiatives are outlined below.

### **Small Group Teaching**

Small group teaching followed automatically from the separation of the Sports Technology students from the Manufacturing Engineering students. However Parsons (2003) had reported that small group teaching was one of the factors in enabling her students to overcome high failure rates and we were therefore hopeful that it would also have a positive effect here.

### **A Computer Algebra System**

There were several reasons for introducing a computer algebra system. The students on the course had widely varying prior qualifications in mathematics. Able students would learn a new skill. Less able students would use Matlab to solve mathematical problems, which may otherwise be beyond them. All the students, regardless of ability, would use Matlab to avoid tedious and time-consuming calculations, thus allowing more time for interpretation of answers.

### **Group Projects**

The ability to function as part of a team is a crucial skill for engineers and the introduction of an assessed group project allows students to learn the skill of teamwork. Moreover, group work provides the chance for students to learn from each other through discussion. However, there can be drawbacks; some students may not do their share of the work. How should the project be marked? Should all members of the team obtain the same mark or should one allocate marks on the basis of each person's contribution? MacBean et al (2001) discuss the above points and provide an overview of the advantages and disadvantages of introducing group work



in undergraduate mathematics. We decided that the advantages to be gained, from students learning to work as part of a team, outweighed the potential disadvantages with the assessment which we believed could be overcome.

### **Sports Examples**

It was decided that applications of mathematics in sport would be used, wherever possible, in order that the students would understand that the mathematics being taught was relevant to their degree. Moreover the group projects would be centred on applications of mathematics in sport. It has been noted (Yates, 2003) that a sufficient supply of discipline related problems is one of the factors leading to success in teaching mathematics to non-specialists.

In addition to these new initiatives, it should be noted that the syllabus for the course was changed to take account of the lowered entry requirement in mathematics.

## **Implementation of the New Initiatives**

This section describes how the four new initiatives were implemented.

### **Small Group Teaching**

In the year 2003–4 there were 34 first year students registered on the Sports Technology course. These students were taught for 3 hours per week over two semesters. The mathematics module was a double module and replaced the two single modules which had run in previous years. Two hours were set aside for traditional mathematics teaching and this time was used for the lecturer to introduce new mathematics and for the students to spend some time doing exercises by hand. The lecturer quickly got to know all the students and any students who started missing classes or coursework assignments or who were not performing well were contacted.

### **A Computer Algebra System**

It was decided that the students should learn Matlab as part of the module. Matlab was chosen as the preferred system as lecturers in the Sports Technology Department required students to use this for some of their final year projects.

It was implemented into the module as follows. Of the three hours per week allocated for the module, one hour was set aside for Matlab work. The students worked in a computer laboratory, with 20 computers. As there were 34 students and only 20 computers, the students were given the option of working in pairs or of splitting the group in two. They decided to opt for the former. In the event the laboratory session normally ran for 90 minutes, with students being free to leave at the end of 50 minutes, but the lecturer and, a postgraduate student assistant stayed on to provide help to those students who wished to work longer.

We support the viewpoint that if a computer algebra system is introduced, it is important to include it in the assessment process. As MacBean et al (2001:7) state,

“...it is clear from our interviews that unless set work (of any kind...) is assessed students are reluctant to participate”. It was decided that assessment of the Matlab work would be via the group project work.

## Group Projects

Each student was required to take part in two group projects, one at the end of the first semester and one at the end of the second semester. Each was worth 10% of the module mark. The students were asked to form groups of three. All the projects required a significant amount of Matlab work and all were based in a sporting context. The topics covered included windsurfing, pole vault, golf, soccer, parachuting and athletics. Each group of students was required to submit a project report, the Matlab files and a poster describing the work undertaken. It was decided that the students within each group would all receive the same mark, except where it was made clear that one or more members of the group had not contributed equally to the final product.

## Sports Examples

Sports examples were used throughout the course, both in lectures and on tutorial sheets. As mentioned already, the group projects were set in a sporting context. Liaison with the Sports Technology lecturers led to some particularly relevant examples, such as the long jump, which was used in the mathematics module as an example of projectile motion and was also covered as part of the biomechanics module.

## Outcome of the New Initiatives

In this section we discuss the success or otherwise of the initiatives. First we look at attendance, then we discuss motivation of the students and the end of year results.

### Attendance

As stated previously, attendance at mathematics tutorials/lectures had been an area of concern in previous years. In the academic year 2002–3, the Mathematics Education Centre recorded attendance at mathematics tutorials and for 2003–4 we recorded attendance in both lectures and tutorials for the Sports Technology module. Figure 1 compares tutorial attendance data for the years 2002–3 and 2003–4. (Note that some data is missing.) We see immediately that there has been a marked improvement in attendance. The trend in 2002–3 was downwards, with zero attendance in six of the eleven weeks in semester 2. Although there was a falling away from high attendance at the beginning of Semester 1 in 2003–4, it did not fall to zero as in the previous year. The average tutorial attendance was 21% in 2002–3 and 68% in 2003–4. We note here that the form of the tutorials was quite different in the two years under comparison. In 2002–3 the tutorials took the form of a traditional

mathematics tutorial where the students were expected to work through exercises and help would be available to those who required it. In 2003–4 the tutorials took place in a computer laboratory and the students worked through exercises using Matlab. In 2003–4 students were given some time in the two hour lecture period to work through exercises by hand. Attendance at lectures followed a similar pattern to the attendance at the tutorials with an average attendance of 70%. Thus in conclusion we see that the new initiatives led to a marked increase in attendance levels, although there is still room for improvement.

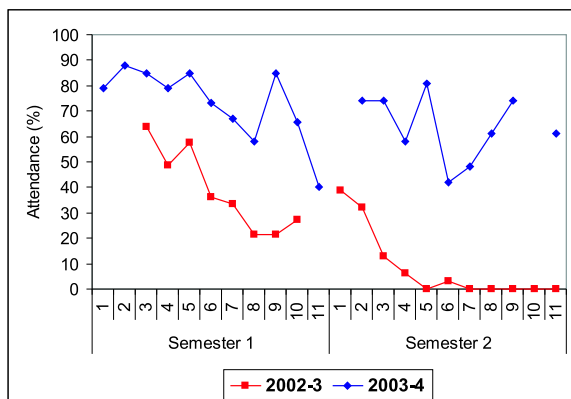


Figure 1. Comparison of Tutorial Attendance

## Motivation

It is not easy to measure the motivation levels of students. Attendance, standard of work and student feedback can all be used as indicators. We have already noted the increased levels of attendance. We also found that the students in 2003–4 worked very hard on the sports based projects and submitted work of a high standard. Figure 2 is a poster from one of the groups, who were investigating the effects of lift, drag and initial velocity on the trajectory of a golf ball. As group projects were not part of the assessment in 2002–3 there can be no comparison with the previous year.

In 2002–3, student feedback was gained informally via students visiting the Mathematics Learning Support Centre. This feedback, from the few students involved, was negative. These students had found the mathematics started at a point beyond their ability and quickly had become very discouraged. In 2003–4, student feedback was obtained through a series of questionnaires and informal discussion. Apart from the University’s standard module feedback questionnaire, the questionnaires were designed by David Marshall, a final year Mathematics student. Under the supervision of the author, Marshall (2004) wrote a dissertation investigating the effects of the new initiatives in the Sports Technology module. Student feedback was positive. We found that most students appreciated the questions set in a sporting context and had enjoyed the group projects. Also the majority of the students felt that learning Matlab was worthwhile, not least because they appreciated that it would be a useful

skill in their later careers. Group projects were particularly highlighted as being useful in getting students to work hard and to meet deadlines—the students did not wish to let down other members of the group.

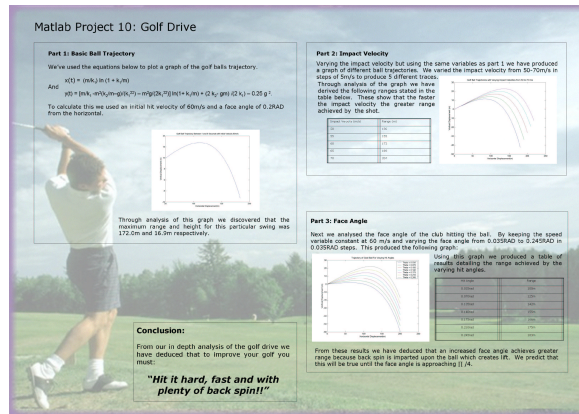


Figure 2. Example of a Group Poster

## Results

Figure 3 compares the coursework and overall module pass rates in 2003–4 with those in 2002–3. In the year 2002–3, the students were set ten computer based tests, which formed their coursework assessment. For 2003–4, the students were set four computer based tests, which formed 50% of the coursework assessment, and they also were set two group projects which formed the other 50%. In both years the students also sat exams at the end of the modules. From Figure 3 we see that there has been a significant improvement with the new module. The module pass rate has increased from 55% to 94% and the coursework pass rate has increased from 65% to 100%. We also found that there was an increase in the level of participation in the assessment process. In 2002–3 the average number of students sitting each test was 87%. In 2003–4 the average sitting tests and handing in projects was 97%.

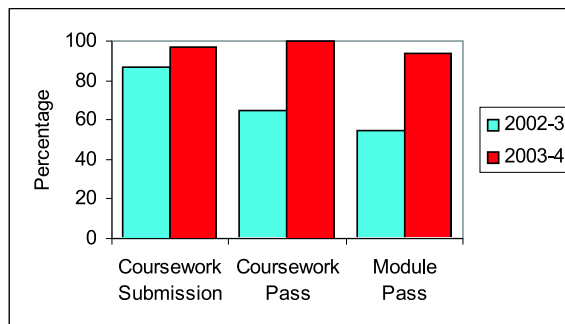


Figure 3. Comparison of Results

However some caution must be exercised over these comparisons. In 2003–4 the mathematics entry requirements for the Sports Technology course were lowered and a more appropriate syllabus was designed for them. Also, in 2002–3, the exam weighting was 80% whereas in 2003–4 it was 60%.

## Conclusions and Some Issues for Others Adopting These Initiatives

We have reported on the implementation of four major changes in the teaching of mathematics to Sports Technology students. These were small group teaching, the introduction of a computer algebra system and group projects and the inclusion of sports related problems into the syllabus. We found that there were significant improvements in attendance levels and in pass rates. Moreover the students appeared more motivated to study the mathematics module.

It is not easy to decide which of the changes introduced had the most impact. All of them played their part. The small group teaching allowed tracking of an individual student's progress and attendance, and intervention could be made when required. The sports based group projects had a significant impact on motivation levels. The introduction of Matlab provided the more able students with a new challenge and also allowed more demanding sports based problems to be set than would otherwise have been the case. The overall picture is that of students who see the relevance of mathematics to their degree and future career and hence are motivated to study it.

For other practitioners wishing to adopt some or all of these changes, there are many issues to consider. Teaching in small groups has distinct advantages, but there are necessarily costs incurred in having more teachers involved. Likewise the introduction of a computer algebra system incurs extra costs—for software licences, computers and the supervision of laboratory sessions. Much time needs to be invested in providing context-based problems. If group work is to be introduced, there needs to be consideration given as to how the work will be marked and in advising students on teamwork. However in the author's opinion, it is worthwhile addressing all of these issues if the effect will be to motivate and enthuse engineering students in their study of mathematics.

## Acknowledgments

My thanks are expressed to Dr Tony Croft, Director, Mathematics Education Centre, Loughborough University, for helpful discussions and suggestions on the work reported in this paper.

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# Visualizations of Derivatives: Zooming on Functions and Vector Fields

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## Abstract

The classical tool to visualize the derivative of a function is an animation where moving secants finally converge to the tangent. The disadvantages of the moving secants are:

1. The adjacent points defining the secants are hard to distinguish.
2. The notion of the derivative as linearizing a function is not made clear.
3. They give a macroscopic view of a microscopic process.

This paper focuses on a microscopic tool to visualize the derivative: zooming into a graph of a function. Increasing magnification finally shows the tangent. This kind of zooming for the tangent is an intuitive zooming because it preserves the aspect ratio.

Other types of zooming with varying aspect ratios are pointed out, especially to visualize the concept of continuity. These zooming techniques for real functions are transferred to vector fields. This approach leads to a graphical understanding of and motivation for vector calculus: curl, line integral, integral formula of Gauß. The ideas of this presentation were shown by M. Kowski [1] in 1997.

## Different kinds of zooming on real functions

### Zooming of first kind—Zooming for differentiability: $\varepsilon \sim \delta$

Zooming into a graph of a function by using a magnifying glass with increasing magnification is an intuitive zooming because the aspect ratio of the  $\varepsilon$ - $\delta$ -box is preserved. Figure 1 and figure 2 show two different states of zooming on a graph of a function together with its tangent. In each figure both frames mark the same part of the graph, each left frame has the original size, while each frame on the right is magnified. In figure 2 the frame in the left picture is so small that it can hardly be seen in the right there is virtually no difference between the function and its tangent. This is a zooming with a fixed relation (aspect ratio) between  $\varepsilon$  and  $\delta$ .

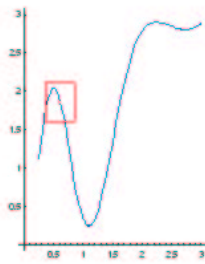


Fig. 1. Zooming of first kind first step

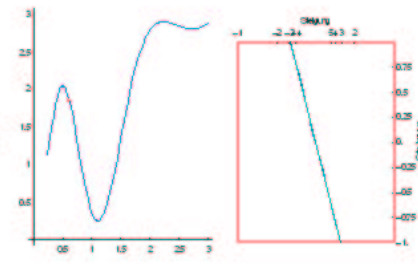


Fig. 2. Zooming of first kind further step

### Zooming of zeroth kind—Zooming for continuity: $\varepsilon \sim \varepsilon_0$

Zooming of zeroth kind means magnifying only the domain and keeping the range fixed. This kind of zooming can be used for checking the continuity by an  $\varepsilon$ - $\delta$ -box. The  $\varepsilon$ -height of the box is given; a suitable  $\delta$ -width of the box must be found, such

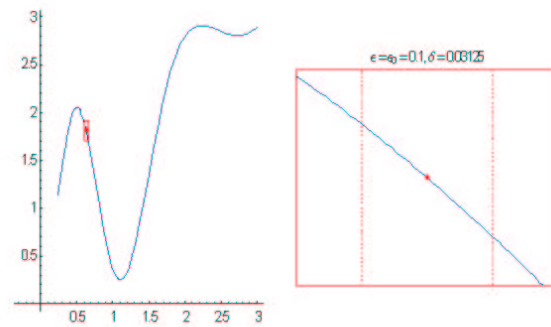


Figure 3. Zooming with fixed range

that the graph of the function will leave the box at its sides. This cannot be checked without a distorted magnification if the given  $\varepsilon$ - $\delta$ -ratio of the box is very small. Therefore the  $\varepsilon$ -height of the box is kept and only the domain is magnified by  $1/\delta$ . The dashed lines in the right picture mark the  $\varepsilon$ - $\delta$ -box of the next step which will be stretched. So the graph gets horizontally stretched, finally it is a horizontal straight line, which corresponds to a constant function. “If the domain is small enough, then the function is nearly constant.” This argument is also used to explain how the integral can be approximated by Riemann sums.

## Linear vector fields

When the notion of derivative is introduced in one dimensional calculus, the students are already very familiar with the linear function  $f(x) = mx + b$ . This is the most



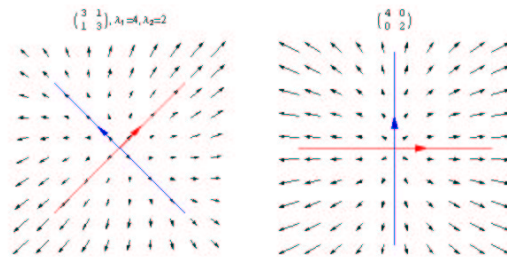
discussed function in school mathematics. A comparable discussion of linear maps is missed even at university. Linear mappings are well studied within the linear algebra course but they are not studied from the calculus point of view, because these examples are too simple.

In this paper we will emphasize linear vector fields. The advantages of considering linear vector fields are:

1. The derivative is a linear map.
2. Curl and divergence can be “seen”.
3. The Gauß formula can be checked without  $\varepsilon$ - $\delta$  by direct calculation.
4. Within simple exercises the integral form of the definition of curl and divergence can be derived by  $\operatorname{div}(L \cdot x) = \operatorname{tr}(L) = \frac{1}{\operatorname{area}(B)} \oint_{\partial B} (L \cdot x) \cdot n \, ds$ .

### Classification of vector fields in the plane

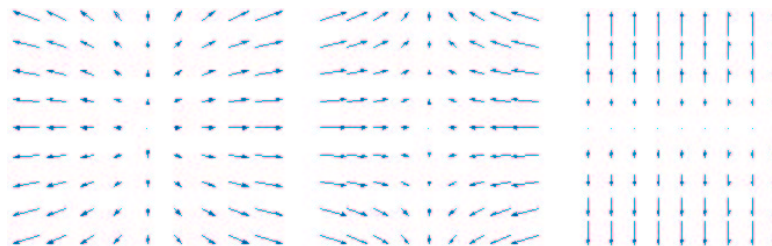
Since each matrix can be decomposed into a symmetric and a skew symmetric part, symmetric and skew symmetric fields are discussed separately.



**Figure 4.** Symmetric vector field together with its diagonalized form

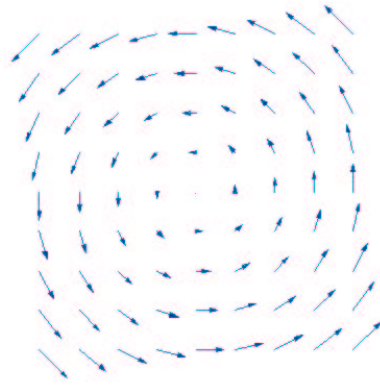
Vector fields which are defined by arbitrary symmetric matrices are obtained graphically just by rotation from the corresponding diagonal matrix (see Figure 4). Therefore we can restrict to diagonal matrices.

Typical examples of symmetric fields are shown in Figure 5.



**Figure 5.** from left to right: both eigenvalues positive, eigenvalues with different sign, rank one

All skew-symmetric fields in the plane are unique up to factor. The matrix is  $\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ . All graphs of them look similar. In Figure 6 the constant  $b$  is positive. Plane skew symmetric fields are invariant under rotations of the plane.

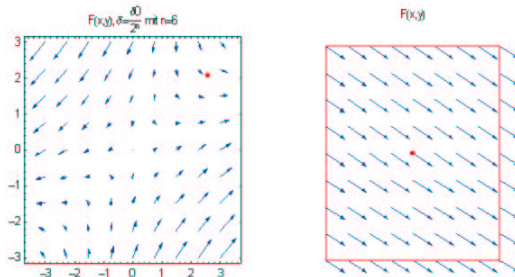


**Figure 6.** Skew symmetric linear field

The divergence of the field is determined only by the symmetric part of a linear field is determining the divergence of the field while the curl is determined only by the skew symmetric part.

## Zooming on vector fields

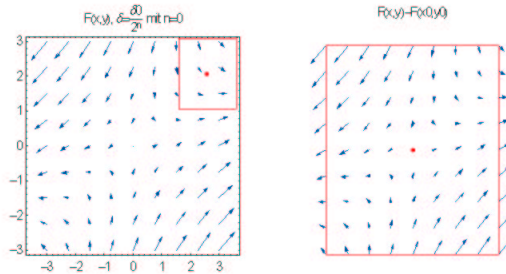
A first attempt for zooming on vector fields is to use a magnifying glass on the plot of the field. In Figure 7 the right box is a highly enlarged small part of the left plot around the marked point of the same linear field.



**Figure 7.** High enlargement with magnification glass

The right box shows a (nearly) constant field and not the expected derivative, which means the same linear field. The reason is that here only the domain but not the (image-) range is magnified. This was the main point of zooming of zeroth kind. This is due to how vector fields are usually plotted: all vectors are plotted with their whole length up to a (nearly zooming independent) scaling factor. Zooming into a function in the same way, would magnify the domain but leave the scale for the

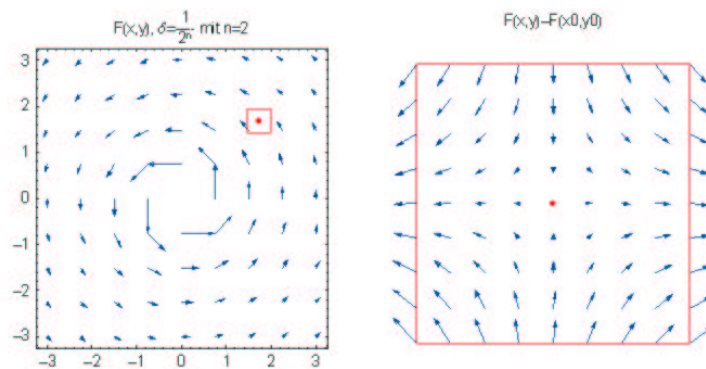
range unchanged. But zooming for differentiability shows the values of the difference  $f(x) - f(x_0)$ . This is the main difference between zooming of zeroth kind and first kind.



**Figure 8.** Zooming of first kind for vector field

Figure 8 shows in the magnified window not the original vector field  $F(x, y)$  but the difference  $F(x, y) - F(x_0, y_0)$ . Like zooming into a straight line only one step reproduces the linear field. Remaining unchanged under zooming of first kind is a necessary condition for a vector field to be linear.

If we apply this zooming procedure of the first kind to non linear vector fields, we see changing plots in the magnified windows, but after only a few steps, the magnified graphics have only slight changes. This is then the **linearization** of  $F$  (see Figure 9).



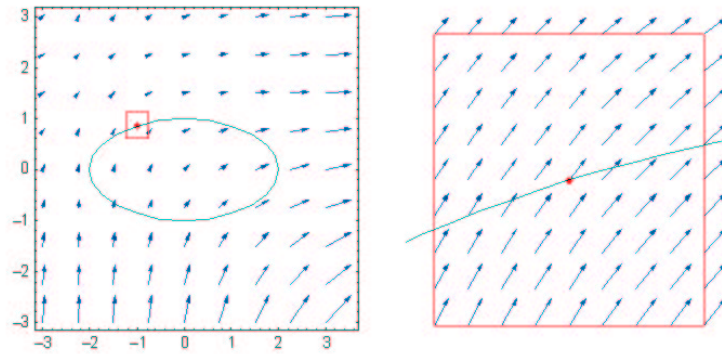
**Figure 9.** Zooming of first kind on the field  $F(x, y) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$ .

Figure 9 also makes clear that the curl of a vector field is not a global property; the curl is a local property. This field looks globally like a swirl but locally after zooming of first kind it has vanishing curl.

## Zooming for line integrals

Finally, we apply zooming of the zeroth kind to the plot of a vector field together with a curve. The curve becomes a straight line and the vector field nearly a constant.

We visualize just the explanations for the line integral: subdivide the curve into small pieces, such that the vector field remains nearly constant and that the curve remains straight, in other words: such that the conditions for the simple calculations of the work as “force times distance” are fulfilled.



**Figure 10.** Zooming for a line integral

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# Calculus for Engineers: A Conceptual Modelling Paradigm

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## Abstract

Advanced mathematics software has been developed and is now commonly available on the Web. Using mathematics software in teaching calculus, there has been a tendency to replace the analytical reasoning of the process by trial-and-error methods using Web materials in some eclectic manner. The question remains if, by such a type of learning process, some basic ideas of teaching calculus is being ignored. Calculus is a hierarchical subject that demands exactness in every detail. In the engineering education process, calculus has a double role: firstly, as a set of methods and results to use in technology; secondly, as the language of science and engineering. Mathematics teaching should aim at learning this language. New pedagogical concepts must be worked out to save basic ideas of teaching calculus. We at EAU (Estonian Agricultural University) have started to develop calculus syllabus using the mathematical models paradigm.

## 1 Teaching calculus using IT

Teaching of mathematics has always been dependent upon technical facilities available for computation. There is a certain evolutionary ‘equilibrium’ between concepts of mathematics teaching and computation. Current rapid advances in computation, linked with a development of mathematical software, should play a major role in the modernisation of mathematics teaching. But it seems that these new computation facilities do not immediately become efficient pedagogical instruments. Due to a rapid development of IT facilities the state of ‘equilibrium’ is going to vanish. The use of modern software in mathematics teaching without proper pedagogical considerations may even cause ‘chaos’ in the calculus-learning environment.

The complexity of software facilities is not an advantage, on the contrary it makes these facilities harder to utilise for pedagogical purposes. When planning to develop software-based calculus curriculum, it is important to realize that once we have made some changes, however small, we are going to have a new learning environment. Starting to use mathematics software in teaching of calculus without analytical reasoning would be a risky matter. Unfortunately, probably due to not

understanding the risk, some mathematics software facilities are put forward to help in the fight against traditional calculus teaching practice. The following text, taken from one of the commercial mathematics software Web materials, exemplifies the situation:

*The use of new technologies developed for the Web allows new forms of educational support to be facilitated, enabling new pedagogical concepts. The trouble with the lecture system is compounded by the fact that our undergraduate courses, for the most part, have been frozen in the past and have become unable to adjust to modern demands. It's time to get rid of it and open the door to some fresh ideas. Increase efficiency by teaching several important ideas simultaneously. Make room for more modern mathematics, moving out of the 19<sup>th</sup> century and into the 21<sup>st</sup>.*

The authors' terms '(learning) technology developed for the Web' and 'new forms of educational support' seem to provide a terminological repository for almost every initiative aimed at changing the way in which higher education is conducted. The success-story phrases above about the possibilities of the software to lead mathematics into the 21<sup>st</sup> century could mislead some people, such as university administrators—and even the engineering professors who might get the idea of advocating the 'just-in-time' philosophy of using only 'fragmental', 'easy' software mathematics. According to this ideology the students would use only software when mathematics is needed in the applications, declining any systematic learning of mathematics.

What is aimed at by mathematical education, and especially by general mathematical education in school and university, is not an efficient mathematical practice supported by currently available computational tools; rather, **it is concerned with the transmission of the bases of 'mathematical culture'**. The values of such a culture are social and, like all other social values, they have a stable core which contributes to shaping our relationship with and interpretation of the surrounding world ([1], p. 246). One of the social values in a community of engineers is 'engineering mathematics'—a rather stable amount of core mathematical knowledge and ideas considered by engineers as natural. Any modified calculus course for engineers must be in accordance with the existing mathematical culture in engineering. According to the SEFI Mathematics Working Group ([4], p. 7) the mathematical topics of particular importance include:

- fluency and confidence with numbers;
- fluency and confidence with algebra;
- knowledge of trigonometric functions,
- understanding of basic calculus and its application to real-world situations;
- proficiency with the collection, management and interpretation of data.

The SEFI Mathematics Working Group also stresses the importance of using the elements of mathematical modelling in calculus (see [4], p. 49):

- It is important that the exposition of the modelling process should be introduced as early in the curriculum as is reasonable.
- The first models that are presented should be simple, so that the process is not obscured by the complexity of the problem, by concepts in engineering not yet encountered and by notation with which the student is unfamiliar.
- The mathematics used in the first models should be straightforward.
- The models must be realistic.
- The physical situation should be one to which the model has been applied in practice. It is of little value to apply a mathematical model to a situation to which it has never been applied, simply to make a pedagogic point.

It seems that the topics above proposed by the SEFI Mathematics Working Group will guarantee the existing ‘mathematical culture’ for new engineers. Having in mind a calculus renewal process by using modern mathematical software we must have clear answers to the following questions:

*What is the role of calculus in the teaching process for engineering students? Is mathematics only a collection of methods to use, or is it also an essential part of the knowledge that students must acquire during the process of education?*

*Are the teachers of calculus prepared to realize the limitation of using computers in their teaching activities; and are they aware of the necessity to develop the ‘paper-and-pencil’ skills of the students in these circumstances?*

*Are the teachers of calculus prepared to recognize the conceptual trouble concerned with transmitting the mathematical culture of engineers into software-based activities?*

These questions should warn us that careful consideration and investigation must be taken before starting to use a calculus software package. Almost all branches of science and engineering rely on mathematics as a language of description and analysis. *The ability to formulate a mathematical model of a given theoretical problem, to solve the model, and to interpret the solution are the key aspects of development for a student*—as regards mathematics. The syllabus must support this development. But the specific role of mathematics must be considered. **Mathematics is a hierarchical subject which demands exactness in every detail. Calculus has a double role in the education process: firstly, it is an amount of methods and results largely used in engineering; secondly, it also constitutes the language of sciences and technique, and the teaching of mathematics should be orientated to the learning of this language.**

## 2 The project “Calculus for Engineering Students”

In more and more countries there is a concern about the decline in the mathematical ability of new entrants to the university degree programmes. The question arises if the students are able to understand the substance of classical calculus. Teaching and learning mathematics are not easy tasks. A routine image of the calculus classroom features a blackboard covered with scarcely readable formulas and exotic symbols. For some students this image generates fear and anxiety. The declining standard of mathematical knowledge and skills of new students is the source of this anxiety. The feeling is justified, for only a few students come to understand the ideas behind calculus. At the same time we get a clear message from our officials to start to teach computer-based, application-orientated mathematics. How should we organize the calculus teaching in these new circumstances? We decided to add a modelling dimension to our teaching process of calculus and initiated the project *Calculus for Engineering Students*, CES.

The project is a continuation of the project *Calculus for biological students*, which had previously been worked out at Estonian Agricultural University (see [5,6]). Our Institute of Mathematics offers 11 points in mathematics and 3 points in statistics for engineering students. The aim of the CES project is to promote the learning of the concepts of calculus by presenting simple technical (mathematical) models, utilising the interdisciplinary approach. The project was also inspired by the ideas of ‘Realistic Mathematics Education’ (RME). The RME is a pedagogical theory where mathematics is considered a real human activity. Van Reeuwijk [8] provides the following characteristics of RME: ‘real’ world; free productions and constructions; mathematization and interaction; integrated learning strands. Drijvers [2,3] explains these points as follows:

1. *‘Real’ world*

The learning of mathematics starts from problem situations that students perceive as real or realistic. These can be real life contexts, but they can also arise from mathematical situations that are meaningful and natural to the students. The word ‘real’ thus refers to ‘experimentally real’ rather than to ‘real world’. The didactical phenomenology of the topic provides adequate contexts that serve as a start for the learning process.

2. *Free productions and constructions*

Students should have the opportunity to develop their own informal problem-solving strategies that can lead to the construction of solution procedures. The models that they develop will gradually turn into generic models for a class of situations.

3. *Mathematization and interaction*

Organising phenomena by means of progressive *matematization* is important in the learning of mathematics. Usually a distinction is made between two types of mathematization: horizontal mathematization, where the problem



situation is modelled into mathematics or vice versa, and vertical mathematization, aiming at a higher level of mathematical abstraction. *Interaction* among students and between students and the teacher is important in RME, because discussion and co-operation enhance the reflection that is essential for the mathematization process.

#### 4. *Integrated learning strands*

According to the philosophy of RME, different mathematical topics should be integrated into one curriculum. The student should develop an integrated view of mathematics, as well as the flexibility to connect the different sub-domains.

One can add to the ‘Realistic Mathematics Education’ principles the principles of the anthropological approach in didactics. The anthropological approach shares with ‘socio-cultural’ approaches in the educational field the vision that mathematics is seen as the product of a human activity. Mathematical production and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities that arise from the practice of given institutions. As regards the objects of knowledge it take in charge, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meaning of knowing or understanding the object. Theory uses terms such as pragmatic value, epistemic value, the routinisation of techniques etc. For obvious reasons of efficiency, advance of knowledge requires the routinisation of some techniques. This routinisation is accompanied by a weakening of the associated theoretical discourse and by a “naturalisation” of associated knowledge which tends to become transparent, to be considered as “natural”. A technique that has become routine now become “de-mathematicised” for the institution. It is important to be aware of this naturalisation process, because through this process techniques lose their “nobility” and become simple acts.

But for any curriculum development it would be very important to understand the instrumentation process of the educational field, which is the subject of the ergonomic approach. According to Artique [1] in the ergonomic approach the concept of the “instrument” itself is important. Researchers in this domain are familiar with working on a professional learning processes, which take place in technologically complex environments. This theory uses terms such as instrumental genesis, instrumentalisation. Using mathematics software enables educators to organize the learning process more efficiently. It enables the student to switch easily between mathematical representations such as graphs, tables and formulae. This can lead to more integrated and flexible use of these representations that will be perceived as different but are in fact related faces of the same die.

For our calculus curriculum renewal project we are going to consider all these ideas. The main priority must be mathematics and its pedagogical needs. The structural identity of calculus must be saved. The mathematical culture of engineering, that is, the core of mathematical knowledge and ideas considered by engineers as natural, must also be saved. Modern technical facilities must be used carefully so as not to abide completely in the mathematics software package ideology - not to

teach the package ideas instead of calculus. In our opinion, this delicate balance is crucial for the curriculum development.

We are also going to consider the following curriculum renewal principles of a *Calculus Consortium at Harvard University* (see [7]):

1. *Start from scratch. Do not look at the old syllabus, trying to decide which topics can be left out. It is much better to take a blank piece of paper and decide which topics are so central that they must be included.*
2. *Show students what calculus can do, not what it cannot do. In a first-year college level course, we should show students the power of calculus, not the special cases in which it fails.*
3. *Be realistic about the abilities of the students and about the amount of time they will dedicate to calculus. In the past, we taught so much so fast that little understanding was developed. It is far better to teach a few topics well.*

Working out the calculus syllabus at EAU, we shall follow all these ideas.

Yet another important point should be taken into account: we must abandon elitist mathematics. The logic of progress of the world shows that more and more students must learn more and more mathematics. In regard to content, we must agree that ‘less is more’: *less memorisation, less mechanics, but more understanding, more thinking, more relations with Nature*. We hope that students will be able to create simple mathematical models that will help them understand the world in which they live. A large number of the examples and problems that students see in the calculus course are given in the context of the real-world problems.

### 3 Conclusions

1. When designing a calculus curriculum one must carefully seek for the right balance between pedagogical methods and mathematics software ideology.
2. Calculus has a double role in the education process: it is an amount of methods and results to use in a scientific research, but it also constitutes the language of science; and the teaching of mathematics should assist the students in learning this language.
3. The computer-based learning environment may cause an unexpected transmission of the bases of ‘mathematical culture’.
4. The descent in the mathematical abilities of the students is a major obstacle, but also a motivator in the process of calculus teaching.

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# The Graphing Calculator and Linear Algebra

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## Changes in Dutch secondary school mathematics

The topic of this paper is related to changes in secondary school mathematics in the Netherlands.

- The first change to be mentioned is the adoption of Real Mathematics Education at all levels of Dutch secondary schools. A characteristic feature of Real Mathematics Education is that mathematical concepts and methods are first introduced and later on applied within the context of problems from real life or problems from other school subjects such as physics, biology or economics.
- In consequence numerical calculations based on real data have made ground in the secondary school mathematics in the Netherlands at the cost of exact calculations and algebraic manipulations.
- The last important change is that for the last few years this shift has been supported by the use of the graphing calculator, the GC, and an extensive list of mathematical formulas in the upper math classes of secondary school and at the final math exams also.

## First reactions of Delft University of Technology

In the academic year of 2001/2002, Dutch universities met for the first time with a number of first year students who had been educated with Real Mathematics Education, the list of formulas and the graphing calculator. The Math Department of Delft University of Technology responded as follows.

- The traditional self written theoretical syllabi and books for Calculus and Linear Algebra courses were replaced by modern, American, textbooks. Although these books do not entirely follow the ideas of Real Math Education, they offer a lot of applications of mathematics to real life problems and problems from other scientific disciplines.

- The short list of formulas used by the students in previous years remained unchanged.
- The GC was ignored in class and forbidden at examinations. It was thought to be better if students learned the basic principles of Calculus and Linear Algebra without any electronic assistance and, when students were eventually ready for electronic tools, to have them use more advanced tools like a PC with Maple and Matlab right away.

Evaluation of the math courses in that year clearly showed that the new style students felt rather uneasy taking math exams without their graphing calculator. So, when in 2002/2003 the vast majority of the first year students arrived with a GC in their bags, the use of this tool was tolerated at the Calculus and Linear Algebra examinations. However, neither the content of the course, nor the character of the exam problems was changed and most teachers took little notice of the possibilities of the graphing calculator. Because of this, some students were able to use their calculators to gain an unintended advantage on the math exams, for example by creating custom programs.

## The next step: a trial project on the GC and Linear Algebra

The next step was to better integrate the graphing calculator into the curriculum and to cope with students' differences in skills using this tool for university math topics. It was decided that there should be a trial project in the academic year of 2003/2004 to gain experience with the GC and Linear Algebra.

The project should be carried out in a Linear Algebra course and not in a Calculus course for the following two reasons.

- Linear Algebra, more than Calculus, calls for recurrent calculations of the same sort that before long do not add much insight.
- In all Linear Algebra courses (except for the one for Civil Engineering), in contrast with a number of Calculus courses, no exercises with Maple or Matlab are set so that no competition between different electronic tools can arise.

Unfortunately, owing to the pressure of other work of the two project leaders, the project was actually set up only shortly before the beginning of the second semester of 2003/2004. By then the first thing to do was to decide which faculty's Linear Algebra course should be the domain of the project. The choice fell on Aerospace Engineering, because

- the Linear Algebra course for this faculty was programmed in the second semester,
- the mathematics records and the interest for mathematics of the Aerospace Engineering students are known to be on an average level in relation to those of the students of the other faculties,

- this faculty would deliver a substantial number of participating students, over 200, split up into eight Linear Algebra classes,
- and last but not least the university mathematics teacher who coordinated the work of his seven colleagues and himself for this course was very willing to cooperate in the project.

## Design of the project plan

The pilot project aimed at two goals.

- In the first place the students should be made familiar with the possibilities of their GC as a tool for Linear Algebra.
- Secondly, they should use their graphing calculator not only when doing the usual exercises, but also to do exercises that could not, or not easily, be done by hand such as applications of Linear Algebra to problems from real life or other scientific disciplines.

Therefore the following plan was designed.

- The students would be provided with a concise guide on the use of their graphing calculator for Linear Algebra.
- A number of exercises to be worked on with the GC would be incorporated in the usual list of recommended exercises. These exercises would be chosen from those exercises in the Linear Algebra textbook that were already marked by the author, David C. Lay, to be solved using an electronic tool.
- The use of the GC would only be tested at the written exam at the end of the course. This arose because experiences in other courses with students handing in results on home made exercises which might increase their final mark, were not all positive. Teachers were always complaining of catching some students who copied each others result's.
- The students could choose between five normal (not requiring a calculator) exam problems and replacing two out of these five by problems meant to be solved using a graphing calculator. Testing the use of the GC had to take place on a voluntary basis because a small minority of the students was not in the possession of a graphing calculator.
- The conditions for the use of the GC at the Linear Algebra exam should be the same as those for the final secondary school exams. This means that
  - only some specific types of calculators are permitted,
  - no graphic calculator may be connected to another device,
  - a student may not borrow another student's calculator during the exam,
  - students, when answering the GC problems, just as with the other problems, should give the reasoning, the formulas and the intermediate results leading to their answers.

## Realization of the plan

Eventually the time available for preparing the necessary materials turned out to be too short for making guides on the use of all permitted types of graphing calculators. This resulted in the decision to concentrate in this respect only on the TI-83 and the TI-83 Plus that the vast majority of students brought from secondary school. Another consequence of the delayed start of the project was that the eight teachers of the course had little or no time to make the necessary preparations, such as making themselves familiar with the GC and its possibilities for Linear Algebra.

Only a week after the beginning of the course the general written information about the project for the students was ready: just like the concise guide on the TI-83 (Plus) and the selection of exercises from the textbook recommended to be worked with a graphing calculator. In the mean time the students had already worked with the usual list of exercises, so the GC exercises were listed separately, which raised the suggestion of extra work. Nevertheless quite a number of the students seemed to appreciate the project plan and were willing to start the exercises right away.

How the teachers went about with the implementation of the ideas of the project has not been investigated. Casual remarks made clear that there were striking differences between them in this respect. Some of them apparently contented themselves with handing their students the written information about the project and subsequently going on as usual. Others tried to stimulate their students to use the graphing calculator by giving examples and paying some attention to the GC exercises in class.

## Evaluation of students' reactions to the project

In one of the last class meetings of the course the students were asked to fill in a questionnaire with respect to their experiences with and their opinions on the pilot project. Unfortunately many of students did not attend these last meetings and also one teacher forgot to hand out the forms. Eventually there turned out to be 62 respondents out of the total number of circa 120 students who in the weeks before still actively participated in the course.

From the 49 respondents who gave their opinion on the value of the *information given about the capabilities of the graphing calculator* with respect to Linear Algebra

- 3 students indicated the given information was “superfluous”,
- 33 students evaluated the information as “worthwhile”,
- 13 students thought it “only a little bit useful”.

The clarity of *the concise guide on the use of the TI-83 (Plus) with Linear Algebra* was evaluated by 39 of the 62 respondents. This guide was thought to be

- “clear and sufficient” by 24 of them,

- “clear but too brief” by 13 others,
- “unclear” by the remaining 2 students.

Almost half of the respondents (30 out of 62) *used the graphing calculator for Linear Algebra* in class or at home.

A vast majority of these (26 out of 30) used it *to do homework exercises from the usual list*. The member using GC straightaway were the same as those who computed the calculations by hand and then use their GC to check their answers. A good quarter of the respondents (17 students) *worked on exercises from the separate list*, but only 6 of them did five or more of these GC exercises.

*Topics for which the graphing calculator was used* by more than a few of the 30 users of this tool, were:

- determining the (reduced) echelon form of a matrix (by all 30),
- the determinant of a matrix (by 21 of them),
- the inverse of a matrix (by 17),
- the least-squares solution of a matrix equation (by 7),
- calculating the eigenvalues of a matrix (only by 5).

So the GC was mainly put into action for calculations that can be done by using a single command from the matrix menu of the TI-83 (Plus). Calculating eigenvalues for instance can only be done with this calculator by entering the characteristic polynomial of the matrix as a function and inspecting the graph of this function in order to determine its zeros.

A vast majority of the respondents (49 out of 62) thought it would be a good idea if the Linear Algebra course should also pay attention to the considerably better capabilities of the computer system *Maple (on PC) as a tool for Linear Algebra*. Only one third of them (17 students) would like, in addition two examples of the use of Maple in class, exercises requiring the use of Maple in the course of materials. *Complaints* of no guide being available for other graphing calculators than the TI, for instance for the Casio, were raised by 9 students.

The only other remark about the project made by more than one student, (in fact also by 9 respondents) was that the use of the graphing calculator with Linear Algebra calls for more attention from and explanation by the teacher in class.

## The exam

As agreed upon, the exam at the end of the course consisted of five problems not requiring the use of an electronic tool, from which two could be replaced by problems meant to be solved by using a graphing calculator.

One of the latter problems was about the orbit of a comet. The eccentricity of the orbit had to be calculated by applying the least-squares method using the four given pairs of decimal numbers related to different positions of the comet in its orbit.



The other GC problem had no real world context and no given decimal numbers. It was a standard problem about transforming a given quadratic form with three variables into a quadratic form with no cross-term. However, in this case the necessary calculation of eigenvalues could not easily be done by hand and to this purpose the GC should be used.

As would be expected on the basis of the results of the questionnaire, only a few students chose to do one or two of the GC problems. In fact, out of the total number of 240 candidates 27 did so. Of those

- 4 tried to do both GC problems,
- 6 students tackled only the problem about the orbit of a comet,
- 17 candidates chose to do only the problem about the quadratic form.

However, most of them either did all calculations of the GC problem(s) by hand, or failed before they got to any calculation that could have been delegated to the graphing calculator. Leaving these candidates aside, the result is:

- 0 students did both GC problems really using the GC,
- 1 student did only the GC problem on the orbit of a comet and actually used the GC to apply the least-squares method,
- 7 did only the GC problem on the quadratic form and actually used the GC to calculate eigenvalues.

The quality of 6 out of the 8 solutions in question was good or reasonable.

How many students used the graphing calculator when doing the normal exam problems and what they used it for, will never be known because they were not asked to write down anything about it. Considering the results of the questionnaire, probably a considerable number of candidates delegated some calculations to their GC and checked other calculations by it. They may have done so mainly when a (reduced) echelon form was needed.

## Conclusions

As mentioned, one of the goals of the pilot project was that students should be made familiar with the possibilities of the GC as a tool for Linear Algebra. From the results of the questionnaire this goal appears to have been reasonably attained with about 50% of the students, especially with respect to determining the (reduced) echelon form of a matrix.

The other goal was that the graphing calculator should be used not only to do the usual exercises, but also to do exercises that could not or not easily be done by hand such as applications of Linear Algebra to real life problems and problems from other scientific disciplines. The results of the questionnaire and the exam make clear that the measure in which this goal is attained is absolutely insufficient. This failure can probably in large part be traced to the poor prior conditions raised by the delayed start of the preparations for the project.

## Recommendations

It seems to be worthwhile to do a similar project next year with the following changes.

- Preparations should start early so that before the start of the course
  - short guides are made up on all GC types brought by the students from secondary school and
  - the recommended GC exercises are integrated in the list of usual exercises.
- Moreover the teachers of the course should make themselves familiar with the GC in good time and be prepared to explain, in class, both the approach to the GC exercises and the more advanced capabilities of the GC with respect to Linear Algebra.
- In addition, the teacher should give examples on the use of Maple with Linear Algebra, but students should not be invited to hand in results on home made exercises using Maple.

## Final remark

Finally it has to be noted that, even if the proposed changes should turn out to bring great improvements, a full integration of the GC in a Linear Algebra course and its exam will be possible only if every student of the course has a graphing calculator at his or her disposal.