Chapter 1

Propositional Logic

Mathematical logic studies correct thinking, correct deductions of statements from other statements. Let us make it more precise. A fundamental property of a statement is that it may be true or false. Whether a statement is true or false is called its truth value. Logic is a systematic study of how statements can be related in ways that capture their respective truth values and how from statements (assumptions) correctly deduce other statements. Knowing which deductions are "logically" correct, we can determine the truth value not only by looking at the words but by looking at the relationship to other statements.

We do not define what a statement is (we also did not define a set, a point etc.). For the reader's convenience, we only describe, in an intuitive way, what we mean by a statement: By a statement we will understand something which is said about the world, and something which has a truth value.

From elementary statements more complicated ones are built, and the truth value of these statements is then determined by the basic ones. To form more complicated statements we use the following logical connectives:

- it is not the case that; we denote it by $\neg$, and call it the *negation*;
- and; we denote it by $\wedge$, and call it the *conjunction*;
- or; we denote it by $\vee$, and call it the *disjunction*;
- if ... then; we denote it by $\Rightarrow$, and call it the *implication*;
- if and only if; we denote it by $\Leftrightarrow$, and call it the *equivalence*.

1.1 Formal Syntax of Propositional Logic

1.1.1 Definition of a Formula. Given a non-empty set A of *logical variables* (we also call them *elementary statements*, or *propositional variables*). A finite sequence of elements of the set A, of logical connectives and parentheses is called a *propositional formula* (or shortly a *formula*), if it is formed by the following rules:

1. Every logical variable (elementary statement) $a \in A$ is a propositional formula.
2. If α, β are propositional formulas, then so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \Rightarrow \beta)$, and $(\alpha \Leftrightarrow \beta)$.
3. Only sequences that were formed by using finitely many applications of rules 1 and 2, are propositional formulas.

The set of all propositional formulas, that were formed from the logical variables from the set A is denoted by $\mathcal{P}(\mathcal{A})$. □

1.1.2 Remark and Notation. The connective $\neg$ is called *unary*, since it forms a new formula from one formula. The other connectives are called *binary*, since they need two formulas to form a new one.

In what follows, we will always denote logical (propositional) variables by small letters: e.g. a, b, c, $\dots$, x, y, z, $\dots$. Propositional formulas will be denoted by small Greek letters: e.g. α, β, γ, $\dots$, φ, ψ, $\dots$

1.1.3 Convention. We will use two rules about usage of parenthesis:

1. We omit the outward parenthesis. For example, we will write $(\alpha \Rightarrow \beta) \Rightarrow \beta$ instead of $((\alpha \Rightarrow \beta) \Rightarrow \beta)$.
2. We assume that the unary connective $\neg$ "is stronger than" each of the binary ones. Hence, if α and β are formulas then we write $\neg\alpha \Rightarrow \beta$ instead of $(\neg\alpha) \Rightarrow \beta$, $\neg\alpha \vee \beta$ instead of $(\neg\alpha) \vee \beta$, etc. After all, you know such situation in arithmetic. For example, $-2+3$ is interpreted as $(-2)+3$, and not as $-(2+3)$.

1.1.4 Syntactic Tree of a Formula. A *syntactic tree* of a formula φ captures its structure; it is a rooted tree where each vertex which is not a leave is labeled by a logical connective and has either one son if the connective is $\neg$, or two sons if the connective is $\wedge$, $\vee$, $\Rightarrow$, or $\Leftrightarrow$. The leaves are labeled by logical variables.

A syntactic tree is also called *derivation tree*.

The *depth of a formula* is defined as the height of the syntactic tree of the formula.

1.1.5 Subformulas of a Given Formula. A *subformula* of a formula α is any substring of α that is a formula itself. $\square$

We can also say that a subformula of α is any string which corresponds to a subtree of the syntactic tree of α.

1.2 Semantics in Propositional Logic

Now we will be interested in the fact whether a correctly formed formula is either true or false. For this we will use the notion of a *truth valuation*.

1.2.1 Truth Valuations of Formulas.

Definition. Given a nonempty set of logical variables A. A mapping $u\colon \mathcal{P}(A) \to \{0,1\}$ is called a *truth valuation*, if it satisfies the following rules

(1) $u(\neg\alpha) = 1$ if and only if $u(\alpha) = 0$;

(2) $u(\alpha \wedge \beta) = 1$ if and only if $u(\alpha) = u(\beta) = 1$;

(3) $u(\alpha \vee \beta) = 0$ if and only if $u(\alpha) = u(\beta) = 0$;

(4) $u(\alpha \Rightarrow \beta) = 0$ if and only if $u(\alpha) = 1$ and $u(\beta) = 0$;

(5) $u(\alpha \Leftrightarrow \beta) = 1$ if and only if $u(\alpha) = u(\beta)$.

$\square$

Here $u(\alpha) = 1$ means that the formula α is true; and $u(\alpha) = 0$ means that the formula α is false.

1.2.2 Truth Tables. The properties that any truth valuation must have, can also be expressed in terms of the truth tables of the logical connectives. These are:

α	$\neg\alpha$
0	1
1	0

α	β	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \Rightarrow \beta$	$\alpha \Leftrightarrow \beta$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

1.2.3 How Many Different Truth Valuation There Are? The answer depends on the number of logical variables. We shall show that if A has n logical variables then there are 2^n distinct truth valuations. For this the following proposition is the key.

1.2.4 Proposition. Every mapping $u_0 : A \to \{0,1\}$ can be uniquely extended to a truth valuation. It means that there is a unique truth valuation $u : \mathcal{P}(A) \to \{0,1\}$ such that $u_0(a) = u(a)$ for all $a \in A$.

Moreover, two truth valuations $u, v : \mathcal{P}(A) \to \{0,1\}$ coincide if and only if $u(x) = v(x)$ for every logical variable $x \in A$. □

Justification. Consider any formula α and its syntactic tree. Evaluate all logical variables by their values in u_0 (i.e. x has the value $u(x) = u_0(x)$). Each vertex in the syntactic tree has the value given by the connective and the value/s of its son/s. Now, the value of the root of the syntactic tree is the truth value of the whole formula.

Notice that the above justification may be turned to an exact proof if mathematical induction is used on the depth of the formula α.

1.2.5 Corollary. Let A contain n logical variables. Then there exist 2^n distinct truth valuations. □

Justification. The above proposition 1.2.4 tells us that the number of distinct truth valuations is the number of distinct mappings $u_0 : A \to \{0,1\}$. And there are 2^n of them.

Remark. Similarly as we formed truth tables for logical connectives we can also form truth tables for any formula. From the above corollary we know that such a truth table will have 2^n rows provided the formula has n logical variables.

1.2.6 A Tautology, a Contradiction, a Satisfiable Formula. Now we can divide formulas into different groups according to their truth values in all valuations.

Definition.

1. A formula is called a *tautology* provided it is true for all truth valuations.
2. A formula is called a *contradiction* provided it is false for all truth valuations.
3. A formula is *satisfiable* provided there is at least one truth valuation for which the formula is true.

□

Remark. It is evident that a negation of any tautology is a contradiction, and conversely, a negation of a contradiction is always a tautology.

For instance, $a \vee \neg a$, $a \Rightarrow a$ are tautologies, whereas $a \wedge \neg a$ is a contradiction. Every tautology is a satisfiable formula, but there are satisfiable formulas that are not tautologies. Indeed, $a \Rightarrow \neg a$ is such an example.

1.2.7 Tautological Equivalence. Formulas of propositional logic are defined as strings of symbols (see 1.1.1), so two formulas are the same if and only if they are the same as strings. Hence the equality of formulas is a very strict notion; indeed, formulas $a \wedge b$ and $b \wedge a$ are different whereas everybody feels that meaning of the conjunction of two formulas does not depend on their order. So we need a new notion for "equality" of formulas; and the notion is the tautological equivalence. There is the formal definition:

Definition. We say that formulas φ and ψ are *tautologically equivalent* (also *semantically equivalent*), if they have the same value in every truth valuation, i.e. if $u(\varphi) = u(\psi)$ for every truth valuation u.

The fact that φ and ψ are tautologically equivalent is denoted by $\varphi \models\!\!\!\!\dashv \psi$. □

1.2.8 Examples. It is very easy to verify that the following tautological equivalences are valid (indeed, it suffices to form corresponding truth tables):

1. $\alpha \wedge \alpha \models\!\!\!\!\dashv \alpha$, $\alpha \vee \alpha \models\!\!\!\!\dashv \alpha$;
2. $\alpha \wedge \beta \models\!\!\!\!\dashv \beta \wedge \alpha$, $\alpha \vee \beta \models\!\!\!\!\dashv \beta \vee \alpha$ (commutativity of $\wedge$ and $\vee$);
3. $\alpha \wedge (\beta \wedge \gamma) \models\!\!\!\!\dashv (\alpha \wedge \beta) \wedge \gamma$, $\alpha \vee (\beta \vee \gamma) \models\!\!\!\!\dashv (\alpha \vee \beta) \vee \gamma$ (associativity of $\wedge$ and $\vee$);
4. $\alpha \wedge (\beta \vee \alpha) \models\!\!\!\!\dashv \alpha$, $\alpha \vee (\beta \wedge \alpha) \models\!\!\!\!\dashv \alpha$ (absorption of $\wedge$ and $\vee$);
5. $\neg\neg\alpha \models\!\!\!\!\dashv \alpha$ (double negation);
6. $\neg(\alpha \wedge \beta) \models\!\!\!\!\dashv (\neg\alpha \vee \neg\beta)$, $\neg(\alpha \vee \beta) \models\!\!\!\!\dashv (\neg\alpha \wedge \neg\beta)$ (de Morgan's laws);
7. $\alpha \wedge (\beta \vee \gamma) \models\!\!\!\!\dashv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, $\alpha \vee (\beta \wedge \gamma) \models\!\!\!\!\dashv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ (distributivity laws).

If moreover **T** is any tautology and **F** is any contradiction, then

8. $\mathbf{T} \wedge \alpha \models\!\!\!\!\dashv \alpha$, $\mathbf{T} \vee \alpha \models\!\!\!\!\dashv \mathbf{T}$, $\mathbf{F} \wedge \alpha \models\!\!\!\!\dashv \mathbf{F}$, $\mathbf{F} \vee \alpha \models\!\!\!\!\dashv \alpha$;
9. $\alpha \wedge \neg\alpha \models\!\!\!\!\dashv \mathbf{F}$, $\alpha \vee \neg\alpha \models\!\!\!\!\dashv \mathbf{T}$.

□

You may notice that some of the above facts are analogous to the properties of the set operations union, intersection, and complement. It is not surprising since these set operations correspond to logical connectives disjunction, conjunction, and negation.

1.2.9 Properties of Tautological Equivalence. There are other properties that the tautological equivalence has and that are useful if we are looking for a simple formula which is tautologically equivalent to a given one. For this the following two propositions play a crucial role.

Proposition. Tautological equivalence satisfies the following properties: For every formulas α, β and γ

1. $\alpha \models\!\!\!\!\dashv \alpha$;
2. if $\alpha \models\!\!\!\!\dashv \beta$ then $\beta \models\!\!\!\!\dashv \alpha$;
3. if $\alpha \models\!\!\!\!\dashv \beta$ and $\beta \models\!\!\!\!\dashv \gamma$ then $\alpha \models\!\!\!\!\dashv \gamma$.

□

Theorem. Let α, β, γ, and δ be formulas satisfying $\alpha \models\!\!\!\!\dashv \beta$ and $\gamma \models\!\!\!\!\dashv \delta$. Then

1. $\neg\alpha \models\!\!\!\!\dashv \neg\beta$;
2. $(\alpha \wedge \gamma) \models\!\!\!\!\dashv (\beta \wedge \delta)$, $(\alpha \vee \gamma) \models\!\!\!\!\dashv (\beta \vee \delta)$, $(\alpha \Rightarrow \gamma) \models\!\!\!\!\dashv (\beta \Rightarrow \delta)$, $(\alpha \Leftrightarrow \gamma) \models\!\!\!\!\dashv (\beta \Leftrightarrow \delta)$.

□

Justification of the above Proposition and Theorem is straightforward and is left as an exercise.

1.2.10 Remark. A formula was defined, see 1.1.1, as a correctly formed string of logical variables, logical connectives ($\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$), and paranthesis. We could started with only four connectives; indeed, any formula of the form $\alpha \Leftrightarrow \beta$ can be rewritten using the following tautological equivalence

$$\alpha \Leftrightarrow \beta \models\!\!\!\!\dashv (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$$

and we get a tautologically equivalent formula that contains only $\neg$, $\vee$, $\wedge$ and $\Rightarrow$.

Similarly, we can introduce **F** as a new symbol, representing a formula that is false in any truth valuation, so it represents a contradiction. Hence, for example $x \Rightarrow \mathbf{F}$ is a well formed formula, moreover $x \Rightarrow \mathbf{F} \models\!\!\!\!\dashv \neg x$.

1.3 Semantical Consequence

Our main aim of this section is to give a more precise meaning to the concept of "correct reasoning", i.e. how to correctly deduce statements/formulas from a given set of formulas/assumptions. What we mean by it: Given a set of *assumptions*, represented by a set of formulas S, we will be looking for formulas that can be *deduced* from S.

Before doing this we specify what we mean by a *set of formulas is true in a truth valuation.*

1.3.1 Definition. Given a truth valuation u and a set of formulas S. We say that S is *true* in u, (or that S is satisfied in u), if every formula from S is true in u. In other words, $u(\varphi) = 1$ for all $\varphi \in S$.

A set of formulas is said to be *satisfiable* if it is true in at least one truth valuation. Otherwise, it is called *unsatisfiable*. □

We will write $u(S) = 1$ whenever S is a set of formulas, u a truth valuation such that S is true in u.

Example. For instance, the set $\{a \Rightarrow b, \neg b\}$ is true for u where $u(a) = 0 = u(b)$, so it is satisfiable. On the other hand, the set $\{a, a \Rightarrow b, \neg b\}$ is unsatisfiable.

1.3.2 Semantical Consequence.

Definition. We say that a formula φ is a *semantical consequence* of a set of formulas S, (or also that φ is an *entailment* of the set S, or that φ *semantically follows* from the set S), provided φ is true for every truth valuation u for which the set S is true.

The fact that formula φ is a semantical consequence of the set S is denoted by $S \models \varphi$. □

Convention. If S is a one element set, e.g. $S = \{\alpha\}$, and $S \models \varphi$, then we write $\alpha \models \varphi$ instead of $\{\alpha\} \models \varphi$.

If S is the empty set $\emptyset$, and $S \models \varphi$, then we write $\models \varphi$ instead of $\emptyset \models \varphi$.

Remarks. 1. It is easy to notice that $S \models \varphi$ if and only if for every valuation u it holds that $u(S) \leq u(\varphi)$.

2. Let us observe that one can verify semantical consequences using truth tables; indeed, we first form truth tables for all formulas in S and for the formula φ. Then we look at all the rows where all formulas from S have 1. In all these rows the formula φ must have 1 as well.

1.3.3 Examples. Let us give couple of examples; they are entailments that are commonly used in many "real life deductions".

For all formulas α, β we have

1. $\{\alpha, \alpha \Rightarrow \beta\} \models \beta$;
2. $\{\alpha \Rightarrow \beta, \neg\beta\} \models \neg\alpha$;
3. $\{\alpha, \neg\alpha\} \models \beta$.

The justification of the above examples is straightforward.

1.3.4 Properties of Semantical Consequence. Let us state several properties that semantical consequences have.

Proposition. For every two formulas φ, ψ we have

- $\varphi \models\mathrel{\mkern-3mu}\dashv \psi$ if and only if $\varphi \models \psi$ and $\psi \models \varphi$.
- We have $\varphi \models \psi$ if and only if the formula $\varphi \Rightarrow \psi$ is a tautology.

□

Justification. 1. $\varphi \models\!\!\!\mid \psi$ means that $u(\varphi) = u(\psi)$ for every truth valuation u. So if $u(\varphi) = 1$ then $u(\psi) = 1$, and if $u(\psi) = 1$ then $u(\varphi) = 1$, which proves the first part of the proposition.

2. Assume that $\varphi \models \psi$ and take arbitrary truth valuation u. Then either $u(\varphi) = 1$, or $u(\varphi) = 0$. In the first case, $u(\psi) = 1$ since $\varphi \models \psi$, and hence $u(\varphi \Rightarrow \psi) = 1$. In the latter case, i.e. if $u(\varphi) = 0$, then from the properties of implication $u(\varphi \Rightarrow \psi) = 1$ as well. Hence $\varphi \Rightarrow \psi$ is a tautology.

Assume that $\varphi \Rightarrow \psi$ is a tautology. Then it cannot happen that $u(\varphi) = 1$ and $u(\psi) = 0$ for any truth valuation u. Hence $\varphi \models \psi$.

1.3.5 More Advanced Properties. We state two further properties that are true for semantical consequence. The first one is a base for so called resolution method. The second one is the Deduction Theorem for propositional logic. We will not prove them, the proofs are not difficult and are left to the readers.

Theorem. Let S be a set of formulas and φ a formula. Then

$$S \models \varphi \quad \text{if and only if} \quad S \cup \{\neg\varphi\} \text{ is unsatisfiable.}$$

Deduction Theorem. Let S be a set of formulas, α and β two formulas. Then

$$S \models (\alpha \Rightarrow \beta) \quad \text{if and only if} \quad S \cup \{\alpha\} \models \beta.$$

1.4 Boolean Calculus

Propositional logic has lot of applications, for example in the theory of logical circuits. In many applications it is useful to form new ”operations” capturing the behavior of logical connectives conjunction, disjunction, and negation.

1.4.1 Logical Operations. Given a truth valuation u and two logical variables a and b. Denote $x = u(a)$ and $y = u(b)$. Then the following holds:

$$\begin{aligned} u(a \vee b) &= \max\{u(a), u(b)\} = \max\{x, y\}, \\ u(a \wedge b) &= \min\{u(a), u(b)\} = \min\{x, y\}, \\ u(\neg a) &= 1 - u(a) = 1 - x. \end{aligned}$$

It motivates the following definition of *boolean operations* for $x, y \in \{0, 1\}$:

$$\begin{aligned} x \cdot y &= \min\{x, y\} \quad \text{(product)}, \\ x + y &= \max\{x, y\} \quad \text{(logical sum)}, \\ \overline{x} &= 1 - x \quad \text{(complement)}. \end{aligned}$$

1.4.2 Properties of Logical Operations. Let us reformulate the properties of logical connectives of $\neg$, $\vee$ and $\wedge$ given in 1.2.8 to the properties of boolean operations.

Proposition. For all $x, y, z \in \{0, 1\}$ we have:

1. $x \cdot x = x,\ x + x = x$;
2. $x \cdot y = y \cdot x,\ x + y = y + x$;
3. $x \cdot (y \cdot z) = (x \cdot y) \cdot z,\ x + (y + z) = (x + y) + z$;
4. $x \cdot (y + x) = x,\ x + (y \cdot x) = x$;
5. $x \cdot (y + z) = (x \cdot y) + (x \cdot z),\ x + (y \cdot z) = (x + y) \cdot (x + z)$;

6. $\overline{\overline{x}} = x$;
7. $\overline{x+y} = \overline{x} \cdot \overline{y},\ \overline{x \cdot y} = \overline{x} + \overline{y}$;
8. $x + \overline{x} = 1,\ x \cdot \overline{x} = 0$;
9. $x \cdot 0 = 0,\ x \cdot 1 = x$;
10. $x + 1 = 1,\ x + 0 = x$.

1.4.3 Boolean Functions. To every formula φ with n logical variables $a_1, \ldots, a_n$ one can assign a function $f\colon \{0,1\}^n \to \{0,1\}$ of n variables $x_1, \ldots, x_n$ defined

$$f(x_1, \ldots, x_n) = u(\varphi) \quad \text{for} \quad u(a_i) = x_i, i = 1, \ldots, n.$$

If two formulas α and β are tautologically equivalent, then their corresponding functions are the same.

Definition. A function $f\colon \{0,1\}^n \to \{0,1\}$ is called a *boolean function of n variables*, where n is a natural number. □

Proposition. To every boolean function $f\colon \{0,1\}^n \to \{0,1\}$ there is a formula α which corresponds to f. □

Notice that the above proposition means that any boolean function can be written as an expression of boolean operations. For example, the boolean function f corresponding to the formula $a \Rightarrow b$ can be written as $f(x,y) = \overline{x} + y$ since $a \Rightarrow b \models\!\!\!\dashv \neg a \vee b$.