Chapter 8

Combinatorics

In this lecture we will focus on so called enumerative combinatorics, it means a way how to count the number of certain objects. At first, we introduce two main principles that will help us to solve more complicated tasks.

8.1 Multiplication and Addition Principles

8.1.1 Multiplication Principle. Assume that a certain activity can be divided into k independent consecutive steps. If step 1 can be done in n_1 ways, step 2 can be done in n_2 ways, etc., and step k can be done in n_k ways, then the number of distinct ways the activity can be done is

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k.$$

□

8.1.2 Example. How many distinct binary words of length n there are?

Solution. A binary word of length n is any sequence $a_1\, a_2\, \ldots\, a_n$ where for all i we have $a_i \in \{0, 1\}$. Such n-tuples can be formed as follow: we choose a_1, then a_2, etc, a_n. For each a_i there are 2 possibilities, indeed, either 0 or 1. So there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ different binary words. □

8.1.3 Addition Principle. Assume that we have n sets $A_1, A_2, \ldots, A_n$ pairwise disjoint (which means that for $i \neq j$ it is $A_i \cap A_j = \emptyset$). Further, assume that each set A_i has k_i elements. The number of elements that can be chosen from A_1 or A_2 or ... or A_n is

$$k_1 + k_2 + \ldots + k_n.$$

Notice that it is the same as the number of elements the set $A_1 \cup A_2 \cup \ldots \cup A_n$ has. □

8.1.4 Example. How many ways can we select two different kinds chocolate bars if we have 4 different dark bars, 5 different milk bars, and 3 different white bars?

Solution. Using the multiplication principle, we know that there are $4 \cdot 5$ different choices of one dark and one milk bar, $4 \cdot 3$ different choices of one dark and one white bar, and $5 \cdot 3$ different choices of one milk and one white bar. These choices are pairwise disjoint, so the number of different choices is

$$20 + 12 + 15 = 47.$$

□

8.2 Permutations, Combinations, Variations

8.2.1 Permutations. To permute objects means to change order of the given objects.

Definition. Given n distinct elements $a_1, a_2, \ldots, a_n$. A *permutation* of $a_1, a_2, \ldots, a_n$ is any ordering of elements $a_1, a_2, \ldots, a_n$. $\square$

Recall that a permutation of $a_1, a_2, \ldots, a_n$ can be viewed as a bijective (i.e. one-to-one and onto) mapping from $\{1, 2, \ldots, n\}$ to $\{a_1, a_2, \ldots, a_n\}$.

Proposition. The number of different permutations of elements $a_1, a_2, \ldots, a_n$ equals to $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$. $\square$

Justification. We use the multiplication principle. For the first element we have n possibilities, indeed, any element $a_1, a_2, \ldots, a_n$. For the second element we can now choose one of $n-1$ different elements (not to the one which was chosen as the first one). For the third element we have $n-2$ possibilities, etc. Hence, all together there are

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

distinct permutations. $\square$

8.2.2 Factorial. For $n \geq 1$ the number

$$n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

is called n *factorial* and denoted by $n!$.

For $n = 0$ we define $0! = 1$.

8.2.3 Example. In a shop there are 6 types of chocolate. How many different ways these 6 types could be exhibited in a row?

Solution. Any permutation of $t_1, \ldots, t_n$ (where t_i represents the i-th type) describes one such exhibition. Hence, there are

$$6! = 6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1 = 720$$

different ways.

8.2.4 Example. How many permutations of letters A, B, C, D, E, F contains CDE as a substring?

Solution. Since the letters CDE must be consecutive and in this order, we can assume that CDE is a new symbol, say Y. Then the question is: how many permutations of A, B, F, Y. There are

$$4 \cdot 3 \cdot 2 \cdot 1 = 24$$

such permutations. $\square$

8.2.5 Variations.

Definition. A k*-variation* of n distinct elements $a_1, a_2, \ldots, a_n$ is a sequence of k (distinct) elements of the set $\{a_1, a_2, \ldots, a_n\}$. The number of distinct k-variations is denoted by $P(n, k)$. $\square$

Remark. k-variations are also called k*-permutations*.

Proposition. The number of k-variations of a set of n distinct elements ($k \leq n$), is

$$P(n, k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

$\square$

Justification. The proof is similar to the proof of the number of permutations. Indeed, for the first element we have n distinct possibilities, for the second element we have $n-1$ distinct possibilities, etc., $\dots$, for the k-th element we have $n-k+1$ distinct possibilities. Now, the multiplication principle finishes the argument. $\square$

Example. A password for a credit card contains four distinct digits.

- How many passwords can be formed?
- How many passwords that do not start with 0 can be formed?

Solution.

1) These are 4-variations of ten digits $0, 1, \dots, 9$. Hence, the number of distinct passwords is

$$10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

2) By the multiplication principle, the number of passwords is

$$9 \cdot (9 \cdot 8 \cdot 7) = 4536.$$

Indeed, for the first digit we have nine possibilities (digits $1, \dots, 9$), and this digit is followed by 3-variation of the remaining digits and 0.

8.2.6 Combinations.

Definition. Given a finite set $A = \{a_1, a_2, \dots, a_n\}$ of n distinct elements. An *k-combination* of A is an unordered selection of k elements of A (in other words, an k element subset of A). The number of distinct k-combinations of n element set is denoted by $C(n,k)$. $\square$

8.2.7 Proposition. The number of distinct k-combinations of n element set equals

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\, k!}.$$

$\square$

Justification. There are $P(n,k)$ distinct k-variations of n distinct elements. k-variations that differ only by ordering correspond to the same k-combination. Since there are $k!$ permutations of a k element set, we get

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} = \frac{n!}{(n-k)!\, k!}.$$

$\square$

8.2.8 Binomial Coefficients. Let $k \le n$ be two natural numbers. Then the number

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

is called a *binomial coefficient* (or a *combinatorial number*).

8.2.9 Proposition.

1) For all $n \in \mathbb{N}$ we have $\binom{n}{0} = 1$
2) For all $n \in \mathbb{N}$ we have $\binom{n}{1} = n$.
3) For all $k \le n$, $k, n \in \mathbb{N}$, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

4) For all $k \le n$, $k, n \in \mathbb{N}$, it holds that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

□

Justification. Properties 1), 2), and 3) are easy consequences of the definition of binomial coefficients.

We will show the property 4): We have

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!\,(k-1)!} + \frac{n!}{(n-k)!\,k!} =$$
$$\frac{n!}{(n-k)!\,(k-1)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) = \frac{n!}{(n-k)!\,(k-1)!}\left(\frac{n+1}{(n-k+1)\,k}\right) =$$
$$\frac{(n+1)!}{(n+1-k)!\,k!} = \binom{n+1}{k}.$$

□

Remark. The last property from 8.2.9 is a basis of so called Pascal triangle.

8.2.10 Variations and Combinations with Repetition. If we allow repetitions then the number of k-variations of n elements is

$$n^k.$$

□

Justification. Indeed, every chosen element can be one of the n elements. Since there are k elements to be chosen, the multiplication principle gives the total number n^k. □

If we choose k elements of A where repetition is allowed then the number of combinations is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}.$$

□

Idea of a justification. Let us show the idea on an example. Assume that we should choose strings of length 4 consisting of letters from the set $\{A, B, C, D, X, Y\}$, repetitions are allowed, but the order letters is not important. Hence, $AAAX$, $CXXY$ are examples of such strings, and strings $AAAX$, $AXAA$, $AAXA$ are considered to be the same. How many distinct strings can be formed?

Any such string can be represented as an 9-tuple consisting of five symbols $|$ and four symbols $\cdot$. The symbol $|$ shows the change of a letter (so we have $6-1=5$ of them, indeed, change from A to B, from B to C, from C to D, from D to X, and from X to Y). The symbol $\cdot$ stands for a letter at the respective positions in the list $\{A, B, C, D, X, Y\}$. For instance

$$AAAX \quad \text{is represented by} \quad \cdot\ \cdot\ \cdot\ |\ |\ |\ |\ \cdot\ |$$

Indeed, there are three A's, no B, no C, no D, one X, and no Y. Similarly,

$$CXXY \quad \text{is represented by} \quad |\ |\ \cdot\ |\ |\ \cdot\ \cdot\ |\ \cdot$$

Indeed, there is no A, no B, one C, two X's, and one Y.

Hence, such a string is represented by choosing four $\cdot$ out of nine positions where a $\cdot$ can be placed (or equivalently, by choosing five $|$ out of nine positions where $|$ can be placed). Therefore, the number of distinct string is

$$\binom{4+6-1}{4} = \binom{4+6-1}{6-1}.$$

Generally, we choose subsets of k elements out of a set of $k+n-1$ distinct places, which equals to

$$\binom{n+k-1}{k}.$$

8.2.11 Example. In a shop 6 types of chocolate bars are sold. Three friends come to a shop and each of them buys one chocolate bar. How many ways could that be done if

1) each friend chooses different type of chocolate bars;
2) they may choose the same type of chocolate bars?

Solution.

1) Call the friends A, B, and C. The number of different choices is $\frac{6!}{3!} = 120$, since we choose triples (where the order is important) out of 6 different types and there is no repetition.

2) If repetitions are allowed then there are $6 \cdot 6 \cdot 6 = 216$ possibilities.

8.2.12 Binomial Theorem. Let us recall the binomial theorem.

Theorem. Let n be a natural number. Then for every real numbers x, y it holds that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k.$$

□

8.2.13 Principle of Inclusion and Exclusion. The addition principle deals with the number of elements which a union of pairwise disjoint sets has. But often we need to know the number of elements a union of two sets A and B has even when A and B are not disjoint.

Theorem. For any sets A, B, C we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

□

Justification. The formula for the number of elements that $A \cup B$ has is evident. Indeed, we sum the number of elements of the both sets and subtract the number of elements of their intersection, since they were calculated twice.

The justification for a union of three sets is similar, only tedious and we omit it. □

8.2.14 Proposition. Let A and B be two sets, $|A| = n$, $|B| = k$. Then there are k^n distinct mappings from A to B. □

Justification. The proposition above is an easy consequence of the multiplication principle. Denote $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_k\}$. □

8.2.15 Dirichlet's, Pigeonhole Principle. This principle is an easy observation but applicable in many counting problems.

Theorem (Pigeonhole principle). Let A and B be two sets, $|A| = n$, $|B| = k$. If $n > k$ then there does not exist a one-to-one mapping from A to B. □

Justification. We will use the multiplication principle. Denote $A = \{a_1, a_2, \ldots, a_n\}$. Let us construct an arbitrary mapping $f \colon A \to B$ which could be one-to-one. For $f(a_1)$ we have k different choices, for $f(a_2)$ only $k-1$ (indeed, we cannot use $f(a_1)$), for $f(a_3)$ only $k-2$ choices, etc, for $f(a_k)$ only a single element of B. Since $n > k$, there is $a_{k+1} \in A$ and $f(a_{k+1})$ must be the same as some of $f(a_1), f(a_2), \ldots, f(a_k)$. Hence, f is not one-to-one. □

8.3 Asymptotic Growth of Functions

8.3.1 Symbol $\mathcal{O}$. Let $g(n)$ be a nonnegative function. We say that a nonnegative function $f(n)$ *is* $\mathcal{O}(g(n))$ if there exists a positive constant c and a natural number n_0 such that

$$f(n) \leq c\, g(n) \quad \text{for every } n \geq n_0.$$

$\mathcal{O}(g(n))$ can be considered as a class of nonnegative function $f(n)$:

$$\mathcal{O}(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \leq c\, g(n) \;\; \forall n \geq n_0\}.$$

8.3.2 Symbol Ω. Let $g(n)$ be a nonnegative function. We say that a nonnegative function $f(n)$ *is* $\Omega(g(n))$ if there exists a positive constant c and a natural number n_0 such that

$$f(n) \geq c\, g(n) \quad \text{for every } n \geq n_0.$$

$\Omega(g(n))$ can be considered as a class of nonnegative functions $f(n)$:

$$\Omega(g(n)) = \{f(n) \mid \exists c > 0, n_0 \text{ such that } f(n) \geq c\, g(n) \;\; \forall\, n \geq n_0\}.$$

8.3.3 Remark. We have $f(n)$ is $\Omega(g(n))$ if and only if $g(n)$ is $\mathcal{O}(f(n))$.

8.3.4 Symbol Θ. Let $g(n)$ be a nonnegative function. We say that a nonnegative function $f(n)$ *is* $\Theta(g(n))$ if there exist positive constants c_1, c_2 and a natural number n_0 such that

$$c_1\, g(n) \leq f(n) \leq c_2\, g(n) \quad \text{for every } n \geq n_0.$$

$\Theta(g(n))$ can be considered as a class of nonnegative functions $f(n)$:

$$\Theta(g(n)) = \{f(n) \mid \exists c_1, c_2 > 0, n_0 \text{ such that } c_1\, g(n) \leq f(n) \leq c_2\, g(n) \;\; \forall\, n \geq n_0\}.$$

8.3.5 Remark. $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $\mathcal{O}(g(n))$ and $\Omega(g(n))$.

8.3.6 Notation. Since the symbols $\mathcal{O}, \Omega, \Theta$ represent sets of functions, we write $f(n) \in \mathcal{O}(g(n))$. Some authors prefer the notation $f(n) = \mathcal{O}(g(n))$. If the later notation is used it is necessary to take in mind that the equality sigh used there does not have all the properties as a classical equality has. Similarly for other symbols.

8.3.7 Proposition. $f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$.

8.3.8 Examples.

1. For every $a > 1$ and $b > 1$ we have

$$\log_a(n) \in \Theta(\log_b(n)).$$

2. The logarithm with base 2 is usually denoted by lg, i.e. $\lg(n) = \log_2(n)$. It holds that

$$\lg n! \in \Theta(n \lg n).$$

The second part of the above proposition follows from the following theorem.

8.3.9 Theorem (Gauss). For every $n \geq 1$ it holds that

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$