Homework 2B – solution

1. Consider the following formula of predicate logic:

$$[(\forall y\,(R(x,y)\wedge P(y)))\Rightarrow(\forall x\,\forall y\,Q(x,y))]\wedge(\forall x\,Q(x,a)).$$

Construct its negation and refurmulate it in such a way that the sign $\neg$ is in front of the "atomic" propositional functions only.

We use the fact that $\neg(\alpha\wedge\beta)=\neg\alpha\vee\neg\beta$ and $\neg(\alpha\Rightarrow\beta)=\alpha\wedge\neg\beta$. In addition, we need to exchange the quantifiers $\forall\leftrightarrow\exists$ when negating quantified formulas. In the following derivation, each line is equivalent to the preceding one:

$$\begin{gathered}\neg\Big([(\forall y\,(R(x,y)\wedge P(y)))\Rightarrow(\forall x\,\forall y\,Q(x,y))]\wedge(\forall x\,Q(x,a))\Big)\\ \neg[(\forall y\,(R(x,y)\wedge P(y)))\Rightarrow(\forall x\,\forall y\,Q(x,y))]\vee\neg(\forall x\,Q(x,a))\\ [(\forall y\,(R(x,y)\wedge P(y)))\wedge\neg(\forall x\,\forall y\,Q(x,y))]\vee\neg(\forall x\,Q(x,a))\\ [(\forall y\,(R(x,y)\wedge P(y)))\wedge(\exists x\,\exists y\neg Q(x,y))]\vee(\exists x\,\neg Q(x,a))\end{gathered}$$

2. Write the negations of the following statements

a) $(\forall\varepsilon>0)(\forall\delta>0)(\varepsilon<\delta)$
b) $(\forall\varepsilon>0)(\exists\delta>0)(\varepsilon<\delta)$
c) $(\exists\varepsilon>0)(\forall\delta>0)(\varepsilon<\delta)$
d) $(\exists\varepsilon>0)(\exists\delta>0)(\varepsilon<\delta)$

Which of them are true? Try to prove or disprove by rigorous argumentation.

For (a), the negation says $(\exists\varepsilon>0)(\exists\delta>0)(\varepsilon\geq\delta)$. The negation is true and hence the original statement is false. Proof: Just take $\varepsilon=2$, $\delta=1$. Then it is indeed true that $\varepsilon\geq\delta$.

For (b), the negation says $(\exists\varepsilon>0)(\forall\delta>0)(\varepsilon\geq\delta)$. The original statement is true and hence the negation is false. Proof: Take arbitrary[1] $\varepsilon>0$. We need to find $\delta>0$ such that $\varepsilon<\delta$. Just take $\delta{:}=\varepsilon+1$. (Or $\delta{:}=2\varepsilon$ or whatever.) Then indeed $\varepsilon<\delta$.

For (c), the negation says $(\forall\varepsilon>0)(\exists\delta>0)(\varepsilon\geq\delta)$. The negation is true and hence the original statement is false. Proof: Take arbitrary $\varepsilon>0$. We need to find $\delta>0$ such that $\varepsilon\geq\delta$. Just take $\delta{:}=\varepsilon/2$. Then indeed $\varepsilon<\delta$.

For (d), the negation says $(\forall\varepsilon>0)(\forall\delta>0)(\varepsilon\geq\delta)$. The original statement is true and hence the negation is false. Proof: Just take $\varepsilon=1$, $\delta=2$. Then it is indeed true that $\varepsilon<\delta$.

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[1] The word *arbitrary* gets sometimes misinterpreted if you are not used to reading mathematical texts. If I say that I will prove something for arbitrary x (or if I ask you to do so), what I mean is not that one can choose x at their convenience and do the proof only for this particular instance. Quite the opposite: I am claiming that the argument should work *for any* x.