Homework 3 – solution

1. Find all integer solutions of the equation

a) $12x - 52y = 16$

b) $512x + 355y = 6$

In (a), you might get confused by the minus sign at the y-term. For the first step – performing the Euclid's algorithm, you can just ignore it:

$$\begin{aligned} 52 &= 4 \cdot 12 + 4 \\ 12 &= 3 \cdot 4 \end{aligned}$$

So, $\gcd(52, 12) = 4$. We also immediatelly get $4 = 52 - 4 \cdot 12$, and multiplying by 4, we get

$$16 = 4 \cdot 52 - 16 \cdot 12.$$

So, one of the solutions is $x = -16$, $y = -4$. By our theorem, we get all the others as $x = -16 - 13k$, $y = -4 - 3k$, $k \in \mathbb{Z}$. Actually, we can slightly simplify it by shifting $k \mapsto -(k+1)$ as $x = -3 + 13k$, $y = -1 + 3k$, $k \in \mathbb{Z}$.

Let's have a look on (b) starting again with the Euclid's algorithm

$$\begin{aligned} 512 &= 1 \cdot 355 + 157 \\ 355 &= 2 \cdot 157 + 41 \\ 157 &= 3 \cdot 41 + 34 \\ 41 &= 1 \cdot 34 + 7 \\ 34 &= 4 \cdot 7 + 6 \\ 7 &= 1 \cdot 6 + 1 \\ 6 &= 6 \cdot 1 \end{aligned}$$

So, $\gcd(512, 355) = 1$, which might have been obvious (since 512 is a power of two and 355 is odd), but we use this also to solve the equation $512x_0 + 355y_0 = 1$.

$$1 = 7 - 6 = -34 + 5 \cdot 7 = 5 \cdot 41 - 6 \cdot 34 = -6 \cdot 157 + 23 \cdot 41 = 23 \cdot 355 - 52 \cdot 157 = -52 \cdot 512 + 75 \cdot 355$$

Multiplying this by 6, we get

$$6 = 312 \cdot 512 + 450 \cdot 355.$$

So, we have got a solution $x = 312$, $y = 450$. By theorem from the lecture, we get all the other solutions as $x = -312 + 355k$, $y = 450 - 512k$, $k \in \mathbb{Z}$. Shifting $k \mapsto k + 1$, we can simplify it to $x = 43 + 355k$, $y = -62 - 512k$, $k \in \mathbb{Z}$.

By the way, we could have computed the last part in a simpler way. Notice that during the Euclid's algorithm, the number 6 appeared as one of the remainders. So, we could have directly computed

$$6 = 34 - 4 \cdot 7 = -4 \cdot 41 + 5 \cdot 34 = 5 \cdot 157 - 19 \cdot 41 = -19 \cdot 355 + 43 \cdot 157 = 43 \cdot 512 - 62 \cdot 355.$$

So, we directly get the particular solution $x = 43$, $y = -62$ and hence the general one is $x = 43 + 355k$, $y = -62 - 512k$, $k \in \mathbb{Z}$.