Homework 4B – solution

1. Find all integer solutions of the congruence

a) $49x \equiv 14 \pmod{35}$ b) $41x \equiv 12 \pmod{501}$

Both problems can be solved using Euclid's algorithm. Regarding the first one, it is actually easier to simplify it using the rules for congruences. We will show both solutions.

First, using Euclid's algorithm. We are solving the equation $14 = 49x + 35k$. Performing Euclid, we get:

$$\begin{aligned} 49 &= 1 \cdot 35 + 14 \\ 35 &= 2 \cdot 14 + 7 \\ 14 &= 2 \cdot 7 \end{aligned}$$

So, $\gcd(49, 35) = 7$ and $7 = 35 - 2 \cdot 14 = -2 \cdot 49 + 3 \cdot 35$. Hence, $14 = -4 \cdot 49 + 6 \cdot 35$. So, one of the solutions is $x = -4$, all the others are given by $x = -4 + 5l$, $l \in \mathbb{Z}$ (as $5 = 35/7$).

The other solution goes as follows (each line is equivalent to the preceding one):

$$\begin{aligned} 49x &\equiv 14 \pmod{35} \\ 14x &\equiv 14 \pmod{35} \\ 7x &\equiv 7 \pmod{35} \\ x &\equiv 1 \pmod{5} \end{aligned}$$

Here, we directly see all the solutions $x = 1 + 5k$, $k \in \mathbb{Z}$.

In (b), there are probably no obvious ways how to simplify the congruence, so we go for Euclid. We are solving $12 = 41x + 501k$.

$$\begin{aligned} 501 &= 12 \cdot 41 + 9 \\ 41 &= 4 \cdot 9 + 5 \\ 9 &= 1 \cdot 5 + 4 \\ 5 &= 1 \cdot 4 + 1 \end{aligned}$$

So, $\gcd(501, 41) = 1$ and $1 = 5 - 4 = -9 + 2 \cdot 5 = 2 \cdot 41 - 9 \cdot 9 = -9 \cdot 501 + 110 \cdot 41$. Consequently, $12 = 1320 \cdot 41 - 108 \cdot 501$. So, the solution is $x = 1320 + 501l$, $l \in \mathbb{Z}$. We can simplify it to the form $x = 318 + 501l$, $l \in \mathbb{Z}$ or $x = -183 + 501l$, $l \in \mathbb{Z}$.