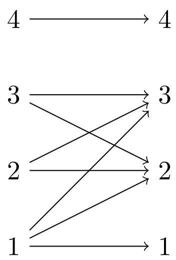
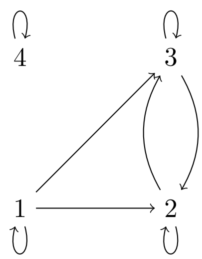
Homework 6B – solution

1. For the following relation on $A = \{1, 2, 3, 4\}$ draw the corresponding graph (using dots and arrows) and decide, whether it is reflexive, symmetric, transitive, or antisymmetric.

The diagrammatic description of R can be drawn in two different ways. Either drawing the set A twice – e.g. two columns and draw lines left-to right – or just once as R is a relation *on* A.





The second option is actually better as it allows to investigate the properties easier. There is a loop at each vertex, so the relation is reflexive. There are pairs of distinct vertices that have only one-way arrow, while there are also pairs of distinct vertices that have both arrows. Hence, the relation is neither symmetric nor antisymmetric. Finally, one can see that the relation is transitive, because any time you go two steps, the first and last vertices are connected by an arrow as well.

2. Consider the following relation on $\mathbb{R}$:

$$x \sim y \quad \Leftrightarrow \quad x^2 = y^2.$$

Show that it is an equivalence. Find the corresponding equivalence classes. Try to find a list of all equivalence classes in which they do not repeat.

We need to check the following properties. Reflexivity: For all $x \in \mathbb{R}$, $x^2 = x^2$ – this is obviously true. Symmetricity: For all $x, y \in \mathbb{R}$, $x^2 = y^2$ implies $y^2 = x^2$ – this is obviously true as well. Transitivity: For all $x, y, z \in \mathbb{R}$, $x^2 = y^2$ and $y^2 = z^2$ implies $x^2 = z^2$. This is true as well. In all three cases, we essentially just used the fact that the equality $=$ has all the properties – being reflexive, symmetric and transitive.

Now, what are the equivalence classes? Well, by definition

$$[x] = \{y \in \mathbb{R} \mid y^2 = x^2\}.$$

Now, the equation $y^2 = x^2$ can actually be simplified as $y = \pm x$. Hence, we clearly have $[x] = [-x] = \{x, -x\}$. On the other hand no other pairs of classes are equal. So, the complete list of equivalence classes consists of

$$[x] = \{x, -x\}, \qquad x \geq 0.$$