Homework 7B – solution

1. Find the inverse of [11] in $(\mathbb{Z}_{36}, \cdot)$.

We look for $[j]$ such that $[11] \cdot [j] = [1]$. That is, $11j \equiv 1 \pmod{36}$. We can do this using Euclid's algorithm:

$$36 = 3 \cdot 11 + 3$$
$$11 = 3 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$

So, $1 = 3 - 2 = -11 + 4 \cdot 3 = 4 \cdot 36 - 13 \cdot 11$. The answer is therefore $[-13]$ or $[23]$. Indeed, $[11] \cdot [23] = [11 \cdot 23] = [253] = [1 + 7 \cdot 36] = [1]$ in $\mathbb{Z}_{36}$.

2. An operation $*$ is defined on the set $\mathbb{R} \times \mathbb{Z}$ by the rule

$$(r, k) * (s, n) = (-rs, k + n).$$

a) Prove that $(\mathbb{R} \times \mathbb{Z}, *)$ is a semigroup.
b) Find a neutral element in $(\mathbb{R} \times \mathbb{Z}, *)$.
c) Find all invertible elements of $(\mathbb{R} \times \mathbb{Z}, *)$.

Since both multiplication in $\mathbb{R}$ as well as addition in $\mathbb{Z}$ are associative, it is straightforward to check that the operation $*$ on $\mathbb{R} \times \mathbb{Z}$ must be associative as well:

$$\begin{aligned}((r,k) * (s,n)) * (t,m) &= (-rs, k+n) * (t,m) = (-(-rs)t, (k+n)+m) \\ &= (-r(-st), k + (n+m)) = (r,k) * (-st, n+m) = (k,k) * ((s,n) * (t,m))\end{aligned}$$

In fact, the operation is also commutative, which is also easy to check, but we did not ask for that.

Now, we are looking for a neutral element. So, we want to find some $e = (x, y)$ such that $(r, k) * e = (r, k) = e * (r, k)$ for every $(r, k) \in \mathbb{R} \times \mathbb{Z}$. This condition means that

$$(r,k) = (x,y) * (r,k) = (-xr, y+k), \quad \text{so} \quad -xr = r,\ y + k = k.$$

This system of equations has a unique solution $x = -1$, $y = 0$. So, the neutral element is given by $e = (-1, 0)$.

Alternatively, you can also guess that the neutral element looks like this and then check it afterwards.

Finally, we are trying to find the inverse of each (r, k). So, we are looking for (x, y) such that $(x, y) * (r, k) = e = (r, k) * (x, y)$. Recall that $e = (-1, 0)$. So, we are trying to solve the following:

$$(-1,0) = (x,y) * (r,k) = (-xr, y+k), \quad \text{so} \quad -xr = -1,\ y + k = 0.$$

This system has a solution if and only if $r \neq 0$. The solution is $x = 1/r$ and $y = -k$.

So, an element (r, k) is invertible if and only if $r \neq 0$. In that case, the inverse is given by $(r, k)^{-1} = (1/r, -k)$.