Homework 13B – solution

1. Find a minimal spanning tree of the weighted graph, where the weights are given by the following matrix

$$\begin{pmatrix}
- & 5 & 9 & 3 & 2 & 5 & 1 \\
5 & - & 18 & 7 & 19 & 1 & 7 \\
9 & 18 & - & 6 & 19 & 10 & 3 \\
3 & 7 & 6 & - & 14 & 8 & 9 \\
2 & 19 & 19 & 14 & - & 7 & 8 \\
5 & 1 & 10 & 8 & 7 & - & 4 \\
1 & 7 & 3 & 9 & 8 & 4 & -
\end{pmatrix}$$

Is the solution you found unique?

We follow the Kruskal's algorithm. We go through all edgest starting from the cheapest to the most expensive and add each to the graph it does not close a cycle. The algorithm ends after adding sixth edge as every tree has exactly $n-1$ edges, where n is the number of vertices (here 7).

In this case, we choose the following edges: $\{1,7\}$, $\{2,6\}$, $\{1,5\}$, $\{1,4\}$, $\{3,7\}$, $\{6,7\}$.

In general, the minimal tree of a weighted graph is not given uniquely. Doing the Kruskal's algorithm, it depends on in which order you consider the edges of equal weight. But in this case it is. Note that constructing it, we essentially never had to do any decisions. We took *all* edges with weight ≤ 4, so there is clearly no other possibility to obtain another tree with the same weight.

2. Consider the directed graph $G=(V,E)$ with $V=\{1,2,\ldots,12\}$ and

$$\begin{aligned}E=\{&(1,4),(3,4),(3,7),(3,9),(3,10),(4,7),(5,2),(5,4),(5,11),(6,1),(6,3),(6,4),(6,5),\\ &(6,7),(6,9),(6,12),(8,9),(8,10),(8,11),(9,11),(12,1),(12,2),(12,3),(12,8)\}\end{aligned}$$

Is the graph G acyclic?

The correct answer is that the graph indeed is acyclic. You might guess it by drawing a picture of it, but that is not a good justification as you might easily miss some cycle.

The most straightforward way to justify that a graph is acyclic is to find its topological sort. So, let's do it. Find the indegree of every vertex in the graph:

vertex	1	2	3	4	5	6	7	8	9	10	11	12
indegree	2	2	2	4	1	0	3	1	3	2	3	1

It's good to check that you did not miss anything by summing the indegrees and checking that it coincides with the total number of edges. The algorithm for finding topological sort T says: put first the vertices with no incoming edges, remove these vertices from the graph and repeat.

So, we start by putting the vertex 6 as the first one in the topological sort and removing it from the graph. We must update the indegrees by lowering the number at vertex 1, 3, 4, 5, 7, 9, 12 as these vertices have an edge from 6:

vertex	1	2	3	4	5	7	8	9	10	11	12
indegree	1	2	1	3	0	2	1	2	2	3	0

So, in the next step, we can remove vertices 5 and 12 and put them to T. We repeat over and over:

vertex	1	2	3	4	5	6	7	8	9	10	11	12	
	2	2	2	4	1	0	3	1	3	2	3	1	$\to T=(6)$
	1	2	1	3	0		2	1	2	2	3	0	$\to T=(6,5,12)$
	0	0	0	2			2	0	2	2	2		$\to T=(6,5,12,1,2,3,8)$
				0			1		0	0	1		$\to T=(6,5,12,1,2,3,8,4,9,10)$
							0				0		$\to T=(6,5,12,1,2,3,8,4,9,10,7,11)$

Since we were able to find the topological sort, the graph must be acyclic.