Positional numeral systems

Theorem. Consider $q \in \mathbb{N}$, $q \geq 2$. Then every number $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{i=0}^{k} a_i q^i$, where $k \in \mathbb{N}_0$, $a_0, \ldots, a_k \in \{0, 1, \ldots, q-1\}$, $a_k \neq 0$.

The number q is called the **base**, the numbers $a_0, \ldots, a_k$ are the **digits**, so the number k represents the **number of digits**. We use the notation $n = (a_k a_{k-1} \cdots a_1 a_0)_q$.

Example. We usually represent numbers in base 10. Here, the digits are $0, 1, 2, \ldots, 9$. For instance, if we write 174, what we mean is *one hundred seventy four*, so more precisely, *one hundred, seven tens, and four ones*, so $174 = 1 \cdot 10^2 + 7 \cdot 10^1 + 4 \cdot 10^0$.

Problem. Write $n = 174$ in base 3.

First, we determine the number of digits. We do this by finding $k \in \mathbb{N}_0$ such that $3^k \leq n < 3^{k+1}$. Here, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 273$. So, the appropriate k is four. Secondly, we want to determine the actual digits. We go from the most significant (the leftmost) to the least (rightmost). Here, the most leftmost digit is a_4, which stands for the *eighty-ones*. So, we ask: How many eighty-ones fit into $n = 174$. The answer is two since $2 \cdot 81 = 162$ (but $3 \cdot 81 = 243$, which is too big already). So, the leftmost digit is two and we are left over with $174 - 162 = 12$, which we need to represent by the other digits. We essentially do the division with remainder. We continue in a similar manner:

$$\begin{aligned}
174 &= \underbrace{2}_{a_4} \cdot \underbrace{81}_{3^4} + 12 \\
12 &= \underbrace{0}_{a_3} \cdot \underbrace{27}_{3^3} + 12 \\
12 &= \underbrace{1}_{a_2} \cdot \underbrace{9}_{3^2} + 3 \\
174 &= \underbrace{1}_{a_1} \cdot \underbrace{3}_{3^1} + \underbrace{0}_{a_0}
\end{aligned}$$

So, we found out that $174 = (20110)_3$.

Algorithm. Input: Numbers $n, q \in \mathbb{N}$, $q > 2$. Output: Expressing n in base q.

find k: $q^k \leq n < q^{k+1}$
for $j = k, k-1, \ldots, 1, 0$ **do**
    find a_j, r: $n = a_j q^j + r$,
    $n \leftarrow r$
return $(a_k, \ldots, a_0)$

Proof of thm. Existence: Basically follows from the algorithm. As an exercise, try to formulate a formal proof using mathematical induction.

Uniqueness: For the sake of contradiction, suppose $n \in \mathbb{N}$ is the smallest number that has two different possible expressions in base q. So, $n = (a_k \cdots a_1 a_0)_q = (b_l \cdots b_1 b_0)_q$. We will study two cases – either $k \neq l$ or $k = l$. In both we are going to derive a contradiction.

So, assume $k = l$. Then $n - q^k$ would also have two different expressions, namely $((a_k - 1)\, a_{k-1} \cdots a_1 a_0)_q$ and $((b_k - 1) b_{k-1} \cdots b_1 b_0)$. This is a contradiction with the assumption that n is the smallest with non-unique expression.

Now, assume $k \neq l$. Without loss of generality, suppose $k > l$. Then using the first expression, we have

$$n = \sum_{i=0}^{k} a_i q^i \geq q^k,$$

but at the same time

$$n = \sum_{j=0}^{l} b_j q^j \leq \sum_{j=0}^{l} (q-1) q^j = (q-1) \frac{q^{l+1} - 1}{q - 1} = q^{l+1} - 1 < q^k.$$

This is obviously a contradiction. □