DMG – exam

Rules

- The time limit is 120 minutes.
- The maximal gain is 80 points.
- Allowed is having a pen, a paper, and an ordinary calculator. It is not allowed to use a cell phone, a computer, or a programmable calculator.
- Every unobvious step in your computations must be explained. If you are genius enough to see the solution right away, you do not have to perform the whole derivation, but you have to convince me that the solution is correct.
- Every computation should be concluded by a clear answer.

Content

It may already not be enough to go through the problems in the homework. Check also everything we did during the exercises on the exercise sheets. Besides being able to do the computations, you should also know all the definitions and you should be able to describe all the algorithms we learned.

The structure of the exam will be as follows:

- 3 easier tasks per 8 points (such as questions on logic, determining gcd, remainders, prime decomposition, generators of a group, order of an element in a group, combinatorics, basic notions in graphs)
- 3 harder tasks per 14 points (Diophantine equation (or a problem related to that), determining properties of a relation, determining properties of an operation / algebraic structure, expressing a number in a given base, performing some graph algorithm)
- 1 describing some algorithm abstractly for 14 points. Here, you have to be as precise as possible! Even a person, who has never heard about this algorithm, should understand how it works and be able to perform it for any input. If you are struggling with explaining it in general, try at least on some example (but you won't get full points).

Sample exam

1. *(8 p.)* Define what a *tautological consequence* means. Prove the following tautological consequence.

$$(a \wedge (b \vee c)) \models (a \vee b)$$

2. *(8 p.)* Define the *order* of an element in a group. Determine the order of [6] in $(\mathbb{Z}_{21}, +)$.

3. *(8 p.)* What is the remainder of $17^{135} + 18^{67} + 19^{256}$ when dividing by 17?

4. *(14 p.)*

a) Consider a triple of natural numbers $a, b, c \in \mathbb{N}$. How can you decide, whether the Diophantine equation $ax + by = c$ has a solution or not?
b) Find all pairs $x, y \in \mathbb{Z}$ that satisfy the equation $168x + 245y = 14$.

5. *(14 p.)* We define a relation R on $\mathbb{N}$ as follows: For any $k, l \in \mathbb{N}$, kRl if and only if $\gcd(k, l) = 2$.

a) Find all $n \in N$ such that $nR2$.
b) Define a *reflexive* relation. Decide, whether R is reflexive.
c) Define a *symmetric* relation. Decide, whether R is symmetric.
d) Define a *transitive* relation. Decide, whether R is transitive.

6. *(14 p.)*

a) Define *Eulerian graph* and *Eulerian cycle*.
b) Formulate a criterion how to decide, whether a graph is Eulerian or not.

Now, consider the graph $G = (V, E)$ with $V = \{1, 2, \ldots, 9\}$ and

$$E = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{3,5\},\{3,7\},\{4,6\},\{4,8\},\{4,9\},\{5,6\},\{5,7\},\{6,8\},\{6,9\}\}$$

c) Use your criterion to decide, whether the graph is Eulerian.
d) If the graph is Eulerian, find the corresponding Eulerian cycle.

7. *(14 p.)* Define a *spanning tree*. Does every graph have a spanning tree? What is the necessary and sufficient condition to have one? Describe the Kruskal's algorithm.

Answers

2) 7; 3) 2; 4a) $\gcd(a, b) \mid c$; 4b) $x = 3 + 35k$, $y = -2 - 24k$, $k \in \mathbb{Z}$; 5) all even numbers, no, yes, no; 6b) all vertices must have an even degree; 6c) yes; 6d) for instance $1-2-3-5-6-8-4-6-9-4-2-5-7-3-1$;