13 Tutorial 13 – January 2nd and 9th, 2018

13.1 There are 150 male students and 40 female students in a class. A delegation of 4 persons will be chosen.

(1) How many ways such a delegation can be chosen?

(2) How many ways a delegation can be chosen if a delegation must contain three male students and one female student?

(3) How many ways a delegation and its spokesperson can be chosen if a delegation must contain three male students and one female student?

Solution.

(1) Because there is no restriction on a delegation, we have altogether $150 + 40 = 190$ students and we choose 4 element subset of it. Hence the number of distinct choices is

$$\binom{190}{4} = 52\ 602\ 165.$$

(2) We choose first a group of 3 male students and independently a female student. So, the number of distinct delegations is

$$\binom{150}{3} \cdot 40 = 551\ 300 \cdot 40 = 22\ 052\ 000.$$

(3) We can calculate the number by two different ways:

a) For each delegation from the part 2) there are 4 possibilities how to choose its spokesman. Hence we have

$$4 \cdot \left(\binom{150}{3} \cdot 40\right) = 4 \cdot 22\ 052\ 000 = 88\ 208\ 000.$$

b) Directly; we first choose a spokesman and then the remaining members of the delegation. We should distinguish between delegations where its spokesman is a male student, and delegations where its spokesman is a female student. Hence, we have

$$150 \cdot 40 \cdot \binom{149}{2} + 40 \cdot \binom{150}{3} = 66\ 156\ 000 + 22\ 052\ 000 = 88\ 208\ 000.$$

13.2 We have 3 cakes with poppy seed and 7 cakes with white cheese. We want to choose 8 cakes. How many ways it can be done if the order is not important?

13.3 How many different ways there are if we want to choose 12 apples from a basket with 20 red apples, 20 green apples, and 20 yellow apples, if at least 3 apples must be chosen of each sort?

13.4 Consider numbers $1, 2, \ldots, n$ where $n \geq 11$. How many ways are there to choose five different numbers from them in such a way that the second largest number number does not exceed 10?

13.5 Prove that for every $n > 0$ it holds that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Solution. The formula can be proved directly using the definition of binomial coefficients, but there is a combinatorial proof as well.

The number $\binom{n}{i}$ is the number of all subsets of the set $\{1, 2, \ldots, n\}$ having i elements. So

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

is the number of all subsets of the set $\{1, 2, \ldots, n\}$. And we know that there are 2^n subsets of it.

13.6 Prove that for any n, k for which $n \geq k > 2$ it holds that

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Solution. We have

$$\binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} = \left(\binom{n-2}{k} + \binom{n-2}{k-1}\right) + \left(\binom{n-2}{k-1} + \binom{n-2}{k-2}\right) =$$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

13.7 In a class of all boys, 13 boys like to play soccer, 17 boys like biking, and 8 boys like hiking. The number of boys who like both soccer and biking is 10, the number of those who like soccer and hiking is 2, and the number of boys who like both biking and hiking is 4. There is one boy who likes all three activities, and 2 boys from the class do not like any of these activities. How many boys there are in the class?

13.8 Suppose that 19 positive even numbers were chosen all of them smaller than 500. Show that there will be at least two numbers whose difference is at most 26.

Solution. Assume that we have chosen 19 even numbers from numbers $2, \ldots, 498$. Consider the smallest chosen number k. If all differences of two numbers chosen are at least 28 then the biggest number must be $k + 18 \cdot 28$, which is $k + 504$ and it is more than 500 (as $k \geq 2$). A contradiction.

13.9 A drawer contains 5 pairs of socks of gray color, 4 pairs of socks of black color, and 4 pairs of socks of dark blue color.

1) How many single socks do we have to take out of the drawer to make sure that we have two socks of the same color?

2) How many single socks do we have to take out of the drawer to make sure that we have two socks of different colors?