DEN Homework 07 – solution

1. a) Given $[a_0, b_0] = [1, 5]$. Check: $f(1) = -1 < 0$, $f(5) = 11 > 0$, opposite signs.
Middle $m_0 = \frac{1}{2}(1+5) = 3$, $f(3) = 1 > 0$, opposite sign compared to $f(1)$, hence a root in $[a_1, b_1] = [1, 3]$.
Middle $m_1 = 2$, $f(2) = -1 < 0$, opposite sign compared to $f(3)$, hence a root in $[a_2, b_2] = [2, 3]$.
We offer the root approximation $m_2 = 2.5$.

b) Formula: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, $f'(x) = 2x - 3$.
$x_0 = 1$, $x_1 = 1 - \frac{f(1)}{f'(1)} = 0$, $x_2 = 0 - \frac{f(0)}{f'(0)} = \frac{1}{3}$.
Bonus: Those who wanted prepared the interation formula $x_{k+1} = x_k - \frac{x_k^2 - 3x_k + 1}{2x_k - 3} = \frac{x_k^2 - 1}{2x_k - 3}$ and saved some calculations.

c) We use the "three-value test".
$f(x_2) = \frac{1}{9} - 1 + 1 = \frac{1}{9} > 0$.
$f(x_2 - \varepsilon) = f\big(\frac{1}{12}\big) = \frac{1}{144} - \frac{36}{144} + 1 = \frac{109}{144} > 0$.
No change of sign yet, so we do not know whether there is a root between $x_2 - \varepsilon$ and x_2.
$f(x_2 + \varepsilon) = f\big(\frac{7}{12}\big) = \frac{49}{144} - \frac{252}{144} + 1 = -\frac{59}{144} < 0$.
The sign changed, so there is a root between x_2 and $x_2 + \varepsilon$. Our approximation is good enough.

2. The scheme: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - x_k + 1}{2x_k - 1} = \frac{x_k^2 - 1}{2x_k - 1}$.
Then $x_0 = 2$, $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = 1$.
Iterations oscillate. It seems that the function is shaped like a valley that does not drop below the x-axis, and the iterations swing left and right in this valley. Since the given function is a polynomial of order two, we guess that it is a parabola above the x-axis, without any root.
Bonus: Using the formula one gets $x_2 = \frac{f(x_1)x_0 - f(x_0)x_1}{f(x_1) - f(x_0)} = \frac{1\cdot 2 - 3\cdot 1}{1-3} = \frac{1}{2}$.
In detail: The line through the points $(2, 3)$ and $(1, 1)$ is given by $y = 3 + \frac{1-3}{1-2}(x - 2) = 2x - 1$.
Choosing $y = 0$ we get the x-intercept: $0 = 2x - 1 \implies x_2 = \frac{1}{2}$.