DEN: ODE – theoretical view: linear equations

Definition.
By a **linear ordinary differential equation of order** n (LODE) we mean any ODE of the form
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x),$$
where $a_{n-1}, \ldots, a_0, b$ are some functions.
This equation is called **homogeneous** if $b(x) = 0$.
Given a linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$, by its **associated homogeneous equation** we mean the equation $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$.

Theorem. (on **existence and uniqueness** for LODE)
Consider a linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$. (L)
If $a_{n-1}, \ldots, a_0, b$ are continuous on an open interval I, then for all $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1} \in I\!R$ there exists a solution to the IVP (L), $y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$ on I and it is unique there.

Theorem. (on **structure of solution set** of homogeneous LODE)
Consider a homogeneous linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$.
If a_i are continuous on an open interval I, then the set of all solutions of this equation on I is a linear space of dimension n.

Definition.
Consider a homogeneous linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$. Assume that a_i are continuous on an open interval I.
By a **fundamental system of solutions** of this equation on I we mean any basis of the space of all solutions of this equation on I.

Definition.
Let y_1, y_2, ..., y_n be $(n-1)$-times differentiable functions. We define their **Wronskian** as
$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem.
Consider a homogeneous linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$,
where a_i are continuous on an open interval I.
Let y_1, y_2, ..., y_n be solutions of this equation on I, let W be their Wronskian.
These functions form a linearly independent set (and thus a fundamental system)
if and only if $W(x) \neq 0$ on I, which is
if and only if there exists $W(x_0) \neq 0$ for some $x_0 \in I$.

Definition.
By a **linear ODE with constant coefficients** we mean any linear ODE for which $a_0(x) = a_0$, $a_1(x) = a_1$, ..., $a_{n-1}(x) = a_{n-1}$ are constant functions.

Definition.
Consider a homogeneous linear ODE with constant coefficients
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0.$$
We define its **characteristic polynomial** as $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$.
We define its **characteristic equation** as $p(\lambda) = 0$. The solutions of this equation are called **characteristic numbers** or **eigenvalues** of the given ODE.

Fact.
Consider a homogeneous linear ODE with constant coefficients
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0.$$
Let λ_0 be its characteristic number. Then $y(x) = e^{\lambda_0 x}$ is a solution of this equation.
If $\lambda_1, \ldots, \lambda_N$ are distinct characteristic numbers of this equation, then $\{e^{\lambda_1 x}, \ldots, e^{\lambda_N x}\}$ is a linearly independent set of solutions.

Fact.
Consider a homogeneous linear ODE with constant coefficients
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0.$$
Let λ_0 be its characteristic number with multiplicity m. Then $e^{\lambda_0 x}, x\,e^{\lambda_0 x}, \ldots, x^{m-1}e^{\lambda_0 x}$ are solutions of this equation and they form a linearly independent set.

Theorem. (on fundamental system for LODE)
Consider a homogeneous linear ODE with constant coefficients
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0.$$
Let λ be its characteristic number of multiplicity m.
(1) If $\lambda = \alpha \in \mathbb{R}$, then $e^{\alpha x}$, $x\,e^{\alpha x}$, $\ldots$, $x^{m-1}e^{\alpha x}$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.
(2) If $\lambda = \alpha \pm \beta i \in \mathbb{C}$, $\beta \neq 0$, then $e^{\alpha x}\cos(\beta x)$, $x\,e^{\alpha x}\cos(\beta x)$, $\ldots$, $x^{m-1}e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$, $x\,e^{\alpha x}\sin(\beta x)$, $\ldots$, $x^{m-1}e^{\alpha x}\sin(\beta x)$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.
(3) The set of functions from (1) and (2) for all characteristic numbers is linearly independent and it forms a fundamental system of the given equation on $\mathbb{R}$.

Theorem. (on **structure of solution set** of linear ODE)
Let y_p be some particular solution of a given linear ODE on an open interval I.
A function y_0 is a solution of this equation on I if and only if $y_0 = y_p + y_h$ for some solution y_h of the associated homogeneous equation on I.
Consequently, if y_h is a general solution of the associated homogeneous equation on I, then $y_p + y_h$ is a general solution of the given equation.

Theorem. (on **guessing a solution** for special right hand-side)
Consider a linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = b(x)$. Assume that $b(x) = e^{\alpha x}[P(x)\cos(\beta x) + Q(x)\sin(\beta x)]$ for some polynomials P, Q, denote $d = \max(\deg(P), \deg(q))$.
Let m be the multiplicity of the number $\lambda = \alpha \pm \beta i$ as a characteristic number of the associated homogeneous equation (we put $m = 0$ if it is not a char. no. at all).
Then there are polynomials $\widetilde{P}, \widetilde{Q}$ of degree at most d such that
$$y(x) = x^m e^{\alpha x}[\widetilde{P}(x)\cos(\beta x) + \widetilde{Q}(x)\sin(\beta x)]$$
is a solution of the given equation on $\mathbb{R}$.
This is called the method of undetermined coefficients.

Simpler forms of incomplete right hand-sides:
- $b(x) = P(x) \implies y(x) = x^m \widetilde{P}(x)$, where m is the multiplicity of $\lambda = 0$;
- $b(x) = P(x)e^{\alpha x} \implies y(x) = x^m \widetilde{P}(x)e^{\alpha x}$, where m is the multiplicity of $\lambda = \alpha$;
- $b(x) = P(x)\cos(\beta x) + Q(x)\sin(\beta x) \implies y(x) = x^m[\widetilde{P}(x)\cos(\beta x) + \widetilde{Q}(x)\sin(\beta x)]$, where m is the multiplicity of $\lambda = \beta i$.

Theorem. (**superposition principle**)
Consider a linear ODE with left hand-side $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$.
Let y_1 be a solution of $L(y) = b_1(x)$ on an open interval I and y_2 be a solution of $L(y) = b_2(x)$ on I.
Then $y_1 + y_2$ is a solution of $L(y) = b_1(x) + b_2(x)$ on I.

Algorithm (variation of parameter method for solving LODE)**.**
Given: equation $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = b(x)$.
1. Using characteristic numbers, find a general solution y_h of the associated homogeneous equation. It has the form $y_h(x) = c_1 \cdot u_1(x) + \ldots + c_n \cdot u_n(x)$.
2. Variation of parameter: Seek a solution of the form $y(x) = c_1(x) \cdot u_1(x) + \ldots + c_n(x) \cdot u_n(x)$.
Unknown functions $c_i(x)$ are found by solving the system of equations

$$\begin{aligned}
c_1'(x)u_1(x) + \ldots + c_n'(x)u_n(x) &= 0\\
c_1'(x)u_1'(x) + \ldots + c_n'(x)u_n'(x) &= 0\\
&\vdots\\
c_1'(x)u_1^{(n-2)}(x) + \ldots + c_n'(x)u_n^{(n-2)}(x) &= 0\\
c_1'(x)u_1^{(n-1)}(x) + \ldots + c_n'(x)u_n^{(n-1)}(x) &= b(x)
\end{aligned}$$

Solve for $c_i'(x)$, integrating them you get $c_i(x)$, substitut these into $y(x) = \sum c_i(x)u_i(x)$.
3. If you take for each $c_i(x)$ one particular antiderivative, then you get one particular solution $y_p(x)$, the general solution is then $y = y_p + y_h$.
If you include "$+C$" when deriving $c_i(x)$, then after substituting them into $y(x) = \sum c_i(x)u_i(x)$ you get the general solution.