

DEN: ODE – theoretical view: systems of equations**Definition.**

By a **system of linear ODEs of order 1 with constant coefficients** we mean a system of the form

$$\begin{aligned}y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x) \\y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x) \\&\vdots \\y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x)\end{aligned}$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in \mathbb{R}$ are coefficients.

An Initial Value Problem (IVP) or Cauchy problem for such a system has initial conditions

$$y_1(x_0) = y_{1,0}, y_2(x_0) = y_{2,0}, \dots, y_n(x_0) = y_{n,0}.$$

The system is called **homogeneous** if $b_i(x) = 0$ for all $i = 1, \dots, n$.

Fact.

Every system of n linear ODEs of order 1 can be transformed via elimination to one linear ODE of order n .

Every system of n linear ODEs of order n_i can be transformed via elimination to one linear ODE of order $\sum n_i$.

Theorem. (on **existence and uniqueness** for systems)

Consider a system of linear ODEs of order 1.

If $b_i(x)$ are continuous on an open interval I , then for every $x_0 \in I$ and all $y_{1,0}, y_{2,0}, \dots, y_{n,0} \in \mathbb{R}$ there exists a solution of the corresponding IVP on I and it is unique.

Fact.

Every linear ODE of order n (and every system of linear ODEs with sum of orders n) can be equivalently transformed to a system of n linear ODEs of order 1.

A system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x) \end{aligned}$$

can be written as $\vec{y}' = A\vec{y} + \vec{b}(x)$,

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is the **matrix of the system**,

$\vec{b}(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$ is the **vector of RHS**, and $\vec{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$ is the unknown, then $\vec{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$.

The system is homogeneous if $\vec{b} = \vec{0}$, where $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$.

Initial conditions are $\vec{y}(x_0) = \vec{y}_0$.

Theorem. (on **structure of solution set** for homogeneous systems)

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$. The set of all solutions of this system on some open interval I is a linear space of dimension n .

Definition.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$.

By a **fundamental system of solutions** of this system on an open interval I we mean any basis of the space of its solutions on I .

If $\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions, then we define its **fundamental matrix** on I by $Y(x) = (\vec{y}_1(x) \ \cdots \ \vec{y}_n(x))$ (an $n \times n$ matrix).

Fact.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$. If $Y(x)$ is its fundamental matrix on I , then a general solution of this system on I is $\vec{y}_h(x) = Y(x) \cdot \vec{c}$ for $\vec{c} \in \mathbb{R}^n$.

Theorem.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$.

Let $\vec{y}_1, \dots, \vec{y}_n$ be solutions of this system on an open interval I .

$\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions of this system on I if and only if $\det(Y(x)) \neq 0$ on I , which is true if and only if $\det(Y(x_0)) \neq 0$ for some $x_0 \in I$.

Definition.

Let $A \in \mathbb{R}^{n \times n}$ be a matrix.

A number λ is called an **eigenvalue** of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$. Vectors \vec{x} with this property are then called **eigenvectors** of A associated with (corresponding to) the eigenvalue λ .

Theorem.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n \times n}$.

If λ_0 is an eigenvalue of A with associated eigenvector \vec{v} , then $\vec{y} = \vec{v}e^{\lambda_0 x}$ is a solution of the given system on \mathbb{R} .

If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of the matrix A , then the corresponding solutions form a linearly independent set.

Fact.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n \times n}$. Let λ_0 be an eigenvalue of A with associated eigenvector \vec{v} .

If λ_0 is a complex number, that is, $\text{Im}(\lambda_0) \neq 0$, then $\text{Re}(\vec{v}e^{\lambda_0 x})$ and $\text{Im}(\vec{v}e^{\lambda_0 x})$ are linearly independent solutions of the given system on \mathbb{R} .

Fact.

Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n \times n}$. Let λ_0 be an eigenvalue of A of multiplicity m with associated eigenvector \vec{v} .

Consider vectors defined as follows:

$$\vec{v}_1 = \vec{v},$$

$$\vec{v}_2 \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_1,$$

$$\vec{v}_3 \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_2,$$

$$\vdots$$

$$\vec{v}_m \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_{m-1}.$$

Then the following functions are solutions of the given system on \mathbb{R} and form a linearly independent set:

$$\vec{y} = \vec{v}_1 e^{\lambda_0 x},$$

$$\vec{y} = \left[\int (\vec{v}_1) dx + \vec{v}_2 \right] e^{\lambda_0 x} = (\vec{v}_1 x + \vec{v}_2) e^{\lambda_0 x},$$

$$\vec{y} = \left[\int (\vec{v}_1 x + \vec{v}_2) dx + \vec{v}_3 \right] e^{\lambda_0 x} = \left(\frac{1}{2} \vec{v}_1 x^2 + \vec{v}_2 x + \vec{v}_3 \right) e^{\lambda_0 x},$$

$$\vdots$$

$$\vec{y} = \left(\frac{1}{(m-1)!} \vec{v}_1 x^{m-1} + \frac{1}{(m-2)!} \vec{v}_2 x^{m-2} + \cdots + \vec{v}_{m-1} x + \vec{v}_m \right) e^{\lambda_0 x}.$$

Theorem. (on **structure of solution set** for systems)

Consider a system of linear ODEs $\vec{y}' = A\vec{y} + \vec{b}(x)$. Let \vec{y}_p be some solution of this system on I . Then \vec{y}_0 is another solution of this system on I if and only if $\vec{y}_0 = \vec{y}_p + \vec{y}_h$ for some solution \vec{y}_h of the system $\vec{y}' = A\vec{y}$ on I .

Thus, if \vec{y}_h is a general solution of the associated homogeneous system on I , then $\vec{y}_p + \vec{y}_h$ is a general solution of the given system on I .

Algorithm (variation of parameters method).

Given: a system $\vec{y}' = A\vec{y} + \vec{b}(x)$.

1. Find a general solution \vec{y}_h of the associated homogeneous system $\vec{y}' = A\vec{y}$:

$$y_{1h}(x) = c_1 u_1(x) + c_2 v_1(x) + \dots,$$

$$y_{2h} = \dots,$$

$$y_{nh}(x) = c_1 u_n(x) + c_2 v_n(x) + \dots$$

2. a) Row variation: We seek solution of the form

$$y_1(x) = c_1(x)u_1(x) + c_2(x)v_1(x) + \dots,$$

$$\vdots$$

$$y_n(x) = c_1(x)u_n(x) + c_2(x)v_n(x) + \dots$$

Unknown functions $c_i(x)$ are found by solving the system of equations

$$c'_1(x)u_1(x) + c'_2(x)v_1(x) + \dots = b_1(x),$$

$$\vdots$$

$$c'_1(x)u_n(x) + c'_2(x)v_n(x) + \dots = b_n(x).$$

From here determine (using e.g. elimination or Cramer rule)

$c'_1(x), \dots, c'_n(x)$, integrating them one gets $c_1(x), \dots, c_n(x)$.

Substitute these into modified y_1, \dots, y_n to get y_{1p}, \dots, y_{np} .

The general solution is $y_i = y_{ip} + y_{ih}$.

b) Vector variation: We write the homogeneous solutions as $\vec{y}_h = Y(x) \cdot \vec{c}$.

We seek solution of the form $\vec{y} = Y(x) \cdot \vec{c}(x)$.

Solve the equation $Y(x) \cdot \vec{c}'(x) = \vec{b}(x)$ for $\vec{c}'(x) = Y(x)^{-1} \vec{b}(x)$.

Integrating by rows get $\vec{c}(x)$ and substitute into $\vec{y}(x) = Y(x) \cdot \vec{c}(x)$.

This yields \vec{y}_p , the general solution is then $\vec{y} = \vec{y}_p + \vec{y}_h$.

Definition.

Assume that the constant function $\vec{y}_s(t) = \vec{y}_0$ is a stationary solution of the system $\vec{y}' = F(t, \vec{y})$. That is, the point \vec{y}_0 is its equilibrium.

We say that this stationary solution is (asymptotically) **stable** if there is $t_s \in \mathbb{R}$ and $\delta > 0$ so that every solution $\vec{y}(t)$ such that $\|\vec{y}(T) - \vec{y}_s(T)\| < \delta$ for some $T \geq t_s$ must necessarily exist on (T, ∞) and converges to \vec{y}_0 as $t \rightarrow \infty$.

We say that \vec{y}_s is **unstable** if it is not stable.

Theorem.

Consider some homogeneous system of linear differential equations $\vec{y}' = A\vec{y}$ with constant coefficients. If all eigenvalues of A have negative real parts, then the equilibrium $\vec{y}_0 = \vec{0}$ is asymptotically stable. Otherwise it is unstable.

Definition.

Let $\vec{y}(t)$, $t \in I$ be a solution to some system of differential equations. We define the **orbit** of this solution as the set

$$\{\vec{y}(t) \in \mathbb{R}^n; t \in I\}.$$

