DEN: ODE – theoretical view: systems of equations

Definition.
By a **system of linear ODEs of order 1 with constant coefficients** we mean a system of the form

$$y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x)$$
$$\vdots$$
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x)$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in \mathbb{R}$ are coefficients.

An Initial Value Problem (IVP) or Cauchy problem for such a system has initial conditions
$y_1(x_0) = y_{1,0}$, $y_2(x_0) = y_{2,0}$, ... , $y_n(x_0) = y_{n,0}$.

The system is called **homogeneous** if $b_i(x) = 0$ for all $i = 1, \ldots, n$.

Fact.
Every system of n linear ODEs of order 1 can be transformed via elimination to one linear ODE of order n.
Every system of n linear ODEs of order n_i can be transformed via elimination to one linear ODE of order $\sum n_i$.

Theorem.   (on **existence and uniqueness** for systems)
Consider a system of linear ODEs of order 1.
If $b_i(x)$ are continuous on an open interval I, then for every $x_0 \in I$ and all $y_{1,0}, y_{2,0}, \ldots, y_{n,0} \in \mathbb{R}$ there exists a solution of the corresponding IVP on I and it is unique.

Fact.
Every linear ODE of order n (and every system of linear ODEs with sum of orders n) can be equivalently transformed to a system of n linear ODEs of order 1.

A system

$$y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x)$$
$$\vdots$$
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x)$$

can be written as $\vec{y}' = A\vec{y} + \vec{b}(x)$,

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is the **matrix of the system**,

$\vec{b}(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$ is the **vector of RHS**, and $\vec{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$ is the unknown, then $\vec{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$.

The system is homogeneous if $\vec{b} = \vec{0}$, where $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$.

Initital conditions are $\vec{y}(x_0) = \vec{y}_0$.

Theorem. (on **structure of solution set** for homogeneous systems)
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$. The set of all solutions of this system on some open interval I is a linear space of dimension n.

Definition.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$.
By a **fundamental system of solutions** of this system on an open interval I we mean any basis of the space of its solutions on I.
If $\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions, then we define its **fundamental matrix** on I by
$$Y(x) = (\,\vec{y}_1(x) \quad \cdots \quad \vec{y}_n(x)\,) \qquad \text{(an } n \times n \text{ matrix)}.$$

Fact.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$. If $Y(x)$ is its fundamental matrix on I, then a general solution of this system on I is $\vec{y}_h(x) = Y(x) \cdot \vec{c}$ for $\vec{c} \in \mathbb{R}^n$.

Theorem.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$, where $A \in \mathbb{R}^{n \times n}$.
Let $\vec{y}_1, \dots, \vec{y}_n$ be solutions of this system on an open interval I.
$\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions of this system on I if and only if $\det(Y(x)) \neq 0$ on I, which is true if and only if $\det(Y(x_0)) \neq 0$ for some $x_0 \in I$.

Definition.
Let $A \in \mathbb{R}^{n \times n}$ be a matrix.
A number λ) is called an **eigenvalues** of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$.
Vectors $\vec{x}$ with this property are then called **eigenvectors** of A associated with (corresponding to) the eigenvalue λ.

Theorem.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n \times n}$.
If λ_0 is an eigenvalue of A with associated eigenvector $\vec{v}$, then $\vec{y} = \vec{v}\,e^{\lambda_0 x}$ is a solution of the given system on $\mathbb{R}$.
If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of the matrix A, then the corresponding solutions form a linearly independent set.

Fact.
Consider a homogeneous system of linear ODEs $\vec{y}' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n \times n}$. Let λ_0 be an eigenvalue of A with associated eigenvector $\vec{v}$.
If λ_0 is a complex number, that is, $\operatorname{Im}(\lambda_0) \neq 0$, then $\operatorname{Re}(\vec{v}e^{\lambda_0 x})$ and $\operatorname{Im}(\vec{v}e^{\lambda_0 x})$ are linearly independent solutions of the given system on $\mathbb{R}$.

Fact.

Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$ with matrix $A \in I\!\!R^{n \times n}$. Let λ_0 be an eigenvalue of A of multiplicity m with associated eigenvector $\vec{v}$.

Consider vectors defined as follows:

$\vec{v}_1 = \vec{v}$,
$\vec{v}_2$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_1$,
$\vec{v}_3$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_2$,

$\quad\vdots$

$\vec{v}_m$ is a solution of $(A - \lambda_0 E_n)\vec{x} = \vec{v}_{m-1}$.

Then the following functions are solutions of the given system on $I\!\!R$ and form a linearly independent set:

$$\vec{y} = \vec{v}_1 e^{\lambda_0 x},$$

$$\vec{y} = \left[\int (\vec{v}_1)\, dx + \vec{v}_2 \right] e^{\lambda_0 x} = (\vec{v}_1 x + \vec{v}_2)e^{\lambda_0 x},$$

$$\vec{y} = \left[\int (\vec{v}_1 x + \vec{v}_2)\, dx + \vec{v}_3 \right] e^{\lambda_0 x} = \left(\tfrac{1}{2}\vec{v}_1 x^2 + \vec{v}_2 x + \vec{v}_3 \right)e^{\lambda_0 x},$$

$$\vdots$$

$$\vec{y} = \left(\tfrac{1}{(m-1)!}\vec{v}_1 x^{m-1} + \tfrac{1}{(m-2)!}\vec{v}_2 x^{m-2} + \cdots + \vec{v}_{m-1} x + \vec{v}_m \right)e^{\lambda_0 x}.$$

Theorem.　(on **structure of solution set** for systems)

Consider a system of linear ODEs $\vec{y}\,' = A\vec{y} + \vec{b}(x)$. Let $\vec{y}_p$ be some solution of this system on I.
Then $\vec{y}_0$ is another solution of this system on I if and only if $\vec{y}_0 = \vec{y}_p + \vec{y}_h$ for some solution $\vec{y}_h$ of the system $\vec{y}\,' = A\vec{y}$ on I.
Thus, if $\vec{y}_h$ is a general solution of the associated homogeneous system on I, then $\vec{y}_p + \vec{y}_h$ is a general solution of the given system on I.

Algorithm　(variation of parameters method)**.**

Given: a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$.

1. Find a general solution $\vec{y}_h$ of the associated homogeneous system $\vec{y}\,' = A\vec{y}$:

$y_{1h}(x) = c_1 u_1(x) + c_2 v_1(x) + \ldots$,

$y_{2h} = \ldots$,

$y_{nh}(x) = c_1 u_n(x) + c_2 v_n(x) + \ldots$

2. a) Row variation: We seek solution of the form

$y_1(x) = c_1(x)u_1(x) + c_2(x)v_1(x) + \ldots$,

$\quad\vdots$

$y_n(x) = c_1(x)u_n(x) + c_2(x)v_n(x) + \ldots$.

Unknown functions $c_i(x)$ are found by solving the system of equations

$c_1'(x)u_1(x) + c_2'(x)v_1(x) + \ldots = b_1(x)$,

$\quad\vdots$

$c_1'(x)u_n(x) + c_2'(x)v_n(x) + \ldots = b_n(x)$.

From here determine (using e.g. elimination or Cramer rule)
$c_1'(x), \ldots, c_n'(x)$, integrating them one gets $c_1(x), \ldots, c_n(x)$.
Substitute these into modified $y_1, \ldots, y_n$ to get $y_{1p}, \ldots, y_{np}$.
The general solution is $y_i = y_{ip} + y_{ih}$.

b) Vector variation: We write the homogeneous solutions as $\vec{y}_h = Y(x) \cdot \vec{c}$.
We seek solution of the form $\vec{y} = Y(x) \cdot \vec{c}(x)$.
Solve the equation $Y(x) \cdot \vec{c}\,'(x) = \vec{b}(x)$ for $\vec{c}\,'(x) = Y(x)^{-1}\vec{b}(x)$.
Integrating by rows get $\vec{c}(x)$ and substitute into $\vec{y}(x) = Y(x) \cdot \vec{c}(x)$.
This yields $\vec{y}_p$, the general solution is then $\vec{y} = \vec{y}_p + \vec{y}_h$.

Definition.
Assume that the constant function $\vec{y}_s(t) = \vec{y}_0$ is a stationary solution of the system $\vec{y}\,' = F(t, \vec{y})$. That is, the point $\vec{y}_0$ is its equilibrium.
We say that this stationary solution is (asymptotically) **stable** if there is $t_s \in I\!\!R$ and $\delta > 0$ so that every solution $\vec{y}(t)$ such that $\|\vec{y}(T) - \vec{y}_s(T)\| < \delta$ for some $T \geq t_s$ must necessarily exist on (T, ∞) and converges to $\vec{y}_0$ as $t \to \infty$.

We say that $\vec{y}_s$ is **unstable** if it is not stable.

Theorem.
Consider some homogeneous system of linear differential equations $\vec{y}\,' = A\vec{y}$ with constant coefficients. If all eigenvalues of A have negative real parts, then the equilibrium $\vec{y}_0 = \vec{0}$ is asymptotically stable. Otherwise it is unstable.

Definition.
Let $\vec{y}(t)$, $t \in I$ be a solution to some system of differential equations. We define the **orbit** of this solution as the set
$$\{\vec{y}(t) \in I\!\!R^n;\ t \in I\}.$$

stable node stable knot	unstable node unstable knot	saddle
stable focus	unstable focus	center