DEN: ODE–transforms

Definition.
Let $a \in I\!R$.
By a **power series with center** a we mean any series of the form $\sum\limits_{k=0}^{\infty} a_k(x-a)^k$,
where $a_k \in I\!R$ and x is a variable.

Theorem.
Let $\sum\limits_{k=0}^{\infty} a_k(x-a)^k$ be a power series.
There exists a number $r \in I\!R_0^+ \cup \{\infty\}$ such that the series converges absolutely for $|x-a| < r$ and diverges for $|x-a| > r$.

This number is called the **radius of convergence** of this series.

Definition.
Consider a function f and a power series $\sum\limits_{k=0}^{\infty} a_k(x-a)^k$.
We say that the series $\sum\limits_{k=0}^{\infty} a_k(x-a)^k$ converges to f uniformly on an interval I if
for every $\varepsilon > 0$ there is $n_0 \in I\!N$ such that

$$\Big| f(x) - \sum_{k=0}^{N} a_k(x-a)^k \Big| < \varepsilon \text{ for } x \in I$$

whenever $N \geq n_0$.

Fact.
Let f be a function such that there is a power series with center a and $r > 0$ so that $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-a)^k$ on $U_r(a)$. Then for every $k \in I\!N_0$ we also have

$$a_k = \frac{f^{(k)}(a)}{k!}.$$

Definition.
Let a function f have derivatives of all orders at a point a.
We define its **Taylor series** with center a as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Finding such a series is called **expanding the given function into a power/Taylor series** (with center a).

Theorem.
Let a function f have derivatives of all orders on some neighborhood $U_r(a)$ with $r > 0$. If there exists $M > 0$ such that $|f^{(k)}(x)| \leq M$ for all $k \in \mathbb{N}_0$ and $x \in U_r(a)$, then $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ on $x \in U_r(a)$.

Fact.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \dots, \ x \in (-1,1);$$
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots, \ x \in \mathbb{R};$$
$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \ x \in \mathbb{R};$$
$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \ x \in \mathbb{R};$$
$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \ x \in (-1,1].$$

Theorem.
Let $a \in I\!\!R$, assume that power series $\sum\limits_{k=0}^{\infty} a_k(x-a)^k = f(x)$, $\sum\limits_{k=0}^{\infty} b_k(x-a)^k = g(x)$ have radii of convergence r_f and r_g.
(i) For all $a, b \in I\!\!R$ we have

$$af(x) + bg(x) = a\Big(\sum_{k=0}^{\infty} a_k(x-a)^k\Big) + b\Big(\sum_{k=0}^{\infty} b_k(x-a)^k\Big) = \sum_{k=0}^{\infty}(aa_k + bb_k)(x-a)^k$$

and this series has radius of convergence $r = \min(r_f, r_g)$.

(ii) We have

$$f(x)\cdot g(x) = \Big(\sum_{k=0}^{\infty} a_k(x-a)^k\Big)\cdot\Big(\sum_{k=0}^{\infty} b_k(x-a)^k\Big) = \sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k} a_i b_{k-i}\Big)(x-a)^k$$

and this series has radius of convergence $r = \min(r_f, r_g)$.

Theorem.
Let a power series $\sum\limits_{k=0}^{\infty} a_k(x-a)^k = f(x)$ have radius of convergence $r > 0$. Then the following are true:
(i) For every $c \in I\!\!R$ we have $f(x-c) = \sum\limits_{k=0}^{\infty} a_k\big((x-c)-a\big)^k = \sum\limits_{k=0}^{\infty} a_k\big(x-(a+c)\big)^k$.
(ii) For every $n \in I\!\!N$ we have $(x-a)^n f(x) = \sum\limits_{k=0}^{\infty} a_k(x-a)^{k+n} = \sum\limits_{k=n}^{\infty} a_{k-n}(x-a)^k$.
(iii) If $\lim\limits_{x\to a}\Big(\dfrac{f(x)}{x-a}\Big)$ converges, then $\frac{1}{(x-a)}f(x) = \sum\limits_{k=1}^{\infty} a_k(x-a)^{k-1} = \sum\limits_{k=0}^{\infty} a_{k+1}(x-a)^k$.
All these series have radius of convergence r_f.

Theorem.
Let a power series $\sum\limits_{k=0}^{\infty} a_k(x-a)^k = f(x)$ have radius of convergence $r > 0$. Then the following are true:
(i) The function f is continuous.
(ii) The function f is differentiable on $U_r(a)$ and on this set we have

$$f'(x) = \sum_{k=1}^{\infty} k\, a_k (x-a)^{k-1}.$$

(iii) The function f has an antiderivative on $U_r(a)$ and

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1} + C.$$

(iv) The function f has derivatives of all orders on $U_r(a)$ and for every $n \in \mathbb{N}$ we have

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdot\ldots\cdot(k-n+1)a_k(x-a)^{k-n} = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k(x-a)^{k-n}.$$

Principle of transformation

Consider a set A with an operation $\circ_A$ and a set B with an operation $\circ_B$. Let T be a 1-1 mapping that satisfies

$$T(x \circ_A y) = T(x) \circ_B T(y)$$

for all $x, y \in A$. Then instead of evaluating $x \circ_A y$ we can use this procedure:

1. We transport the problem to the world B: $T(x)$, $T(y)$.
2. We solve the problem in the world B: $T(x) \circ_B T(y)$.
3. We move the result back to the world A: $T^{-1}\big(T(x) \circ_B T(y)\big)$.

Fact.
Let $a \in I\!R$, let V be the space of functions that can be expanded into a power series with center a. For f define a mapping T by the condition $T(f) = \{a_k\}_{k=0}^{\infty}$ if $f(x) = \sum\limits_{k=0}^{\infty} a_k (x-a)^k$ on some $U_r(a)$.
Take any $f, g \in V$ and assume that $T(f) = \{a_k\}_{k=0}^{\infty}$ and $T(b) = \{b_k\}_{k=0}^{\infty}$. Then the following are true:

(i) $T(\alpha f + \beta g) = \{\alpha a_k + \beta b_k\}_{k=0}^{\infty}$ for all $\alpha, \beta \in I\!R$.

(ii) $T((x-a)f) = \{0, a_0, a_1, a_2, \dots\}$.

(iii) If $f(a) = 0$, then $T\big(\frac{1}{x-a} f\big) = \{a_{k+1}\}_{k=0}^{\infty} = \{a_1, a_2, a_3, \dots\}$.

(iv) $T(f') = \{(k+1)a_{k+1}\}_{k=0}^{\infty} = \{a_1, 2a_2, 3a_3, \dots\}$.