DEN: ODE–Fourier transform

Definition.
Let f be a function defined on an interval $I = [a, a+T)$ for some $a \in I\!R$, $T > 0$.
Its **periodic extension** is defined as the function

$$f(t) = f(t - kT) \qquad \text{for} \;\; t \in [a + kT, a + (k+1)T).$$

Theorem.
Let f be a function that is T-periodic. Let $\omega = \frac{2\pi}{T}$.
If the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$$

converges to $f(x)$ on $I\!R$ uniformly, then necessarily

$$a_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega t)\, dt \text{ pro } k \in I\!N_0,$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t)\, dt \text{ pro } k \in I\!N.$$

Definition.
Let f be a function integrable on an interval $[a, a+T]$.
We define its **Fourier series** as

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big],$$

where $\omega = \frac{2\pi}{T}$ and

$$a_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega t)\, dt \text{ pro } k \in \mathbb{N}_0,$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t)\, dt \text{ pro } k \in \mathbb{N}.$$

We write

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big].$$

Theorem. (Jordan criterion)
Let f be a function that is piecewise continuous on some interval I of length T, assume that it has derivative f' that is piecewise continuous on I.
Consider its periodic extension, call it f again for simplicity.
Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \bigl[a_k \cos(k\omega t) + b_k \sin(k\omega t)\bigr]$. Then for every $t \in I\!R$ we have

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \bigl[a_k \cos(k\omega t) + b_k \sin(k\omega t)\bigr] = \frac{1}{2}\Bigl[\lim_{x\to t^-} \bigl(f(x)\bigr) + \lim_{x\to t^+} \bigl(f(x)\bigr)\Bigr].$$

If, moreover, f is continuous on $I\!R$, then $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \bigl[a_k \cos(k\omega t) + b_k \sin(k\omega t)\bigr] = f$ and the convergence of this series is uniform on $I\!R$.

Fourier series in amplitude-phase form:

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k).$$

Fourier series in complex form:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t},$$

where

$$c_k = \tfrac{1}{T} \int_a^{a+T} f(t) e^{-ik\omega t} dt.$$

Definition.
Let $f(t)$ be a function integrable on $I\!R$. Its **Fourier transform** $\mathcal{F}[f](\omega)$ is defined by the formula

$$\mathcal{F}[f]: \ \omega \mapsto \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt.$$

Often we also write $\mathcal{F}[f] = \hat{f}$.

Theorem.
Let f be a function integrable on $I\!R$ that is continuous with possible exception of finitely many jump discontinuities. If we redefine f as $f(x) = \frac{1}{2}\big(f(x^-) + f(x^+)\big)$ at these points, then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega.$$

Fourier transform:

Dictionary:

$\mathcal{F}[1] = 2\pi\delta(\omega)$;

$\mathcal{F}[\delta(t)] = 1$;

$\mathcal{F}[H(t)] = \pi\delta(\omega) + \frac{1}{i\omega}$;

$\mathcal{F}[\operatorname{sgn}(t)] = \frac{2}{i\omega}$;

$\mathcal{F}[e^{i\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$;

$\mathcal{F}[e^{-\omega_0 |t|}] = \frac{2\omega_0}{\omega_0^2 + \omega^2}$; $(\omega_0 > 0)$

$\mathcal{F}[e^{i\omega_0 t} H(t)] = \frac{1}{\omega_0 + i\omega}$;

$\mathcal{F}[t\, e^{i\omega_0 t} H(t)] = \frac{1}{(\omega_0 + i\omega)^2}$; $(\omega_0 > 0)$

$\mathcal{F}[\sin(\omega_0 t)] = \frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$;

$\mathcal{F}[\cos(\omega_0 t)] = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$.

Grammar:

$\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g]$;

$\mathcal{F}[f(t-a)] = e^{-ia\omega}\hat{f}(\omega)$;

$\mathcal{F}[t\, f(t)] = i\hat{f}'(\omega)$;

$\mathcal{F}[e^{iat} f(t)] = \hat{f}(\omega - a)$;

$\mathcal{F}[f'(t)] = i\omega\hat{f}(\omega)$.

here $\hat{f} = \mathcal{F}[f]$