

DEN: ODE–Laplace transform

Definition.
For f: $[0,\infty) \mapsto \mathbb{R}$ we define its **Laplace transform** $\mathcal{L}\{f(t)\}$ by the formula

$$\mathcal{L}\{f(t)\}:\ s \mapsto \int\limits_0^\infty f(t)e^{-st}dt,$$

assuming that this converges for at least one $s \in \mathbb{R}$.

Notation: $\mathcal{L}\{f(t)\}(s)$, $\mathcal{L}\{f\}$, $F(s)$, alternatively $f(t) \,\widehat{=}\, F(s)$.

Definition.

$$\mathcal{L}_0 = \big\{f(t):\ [0,\infty) \mapsto \mathbb{R};\ f \text{ piecewise continuous and } |f(t)| \le C\,e^{\alpha t} \text{ for some } C,\alpha > 0\}.$$

Definition.
Heaviside function is defined as

$$H(t) = \begin{cases} 1, & t \ge 0; \\ 0, & t < 0. \end{cases}$$

Fact. (dictionary)

- $\mathcal{L}\{e^{\alpha t}H(t)\} = \frac{1}{s-\alpha}$, $s > \alpha$ for all $\alpha \in \mathbb{R}$;
- $\mathcal{L}\{t^n H(t)\} = \frac{n!}{s^{n+1}}$, $s > 0$ for all $n \in \mathbb{N}_0$;
- $\mathcal{L}\{\sin(\omega t)H(t)\} = \frac{\omega}{s^2+\omega^2}$, $s \in \mathbb{R}$ for all $\omega \in \mathbb{R}$;
- $\mathcal{L}\{\cos(\omega t)H(t)\} = \frac{s}{s^2+\omega^2}$, $s \in \mathbb{R}$ for all $\omega \in \mathbb{R}$.

Theorem. (grammar)
Let $f \in \mathcal{L}_0$.
(i) (linearity) $\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ for every $g \in \mathcal{L}_0$ and all $a,b \in I\!R$.
(ii) (change of scale) $\mathcal{L}\{f(at)\} = \frac{1}{a}\mathcal{L}\{f(t)\}\big|_{s/a}$ for every $a > 0$;
(iii) (shift in image) $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{s-a}$ for every $a \in I\!R$;
(iv) (derivative of image) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f(t)\}$ for every $n \in I\!N$;
(v) (derivative of preimage) If $f^{(n)} \in \mathcal{L}_0$, then
$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \ldots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$;
(vi) (integral of preimage) $\mathcal{L}\big\{\int_0^t f(u)\,du\big\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.
(vii) (integral of image) If $\lim\limits_{t\to 0^+}\big(\frac{f(t)}{t}\big)$ converges, then $\mathcal{L}\big\{\frac{1}{t}f(t)\big\} = \int\limits_s^\infty \mathcal{L}\{f(t)\}(q)\,dq$.

Theorem.
Assume that for some $f,g\in\mathcal{L}_0$ we have $\mathcal{L}\{f\}=\mathcal{L}\{g\}$ on some $[s_0,\infty)$. Then $f=g$ up to at most countable set of isolated points.
If, moreover, f and g are continuous from the right everywhere, then $f=g$.

Theorem.
If $F(s)$ is a rational function whose numerator has smaller degree than the denominator, then the inverse $\mathcal{L}^{-1}\{F(s)\}$ exists and can be found using partial fractions decompositon.

Fact.
$\mathcal{L}^{-1}\big\{\frac{1}{s-\alpha}\big\}=e^{\alpha t}$,
$\mathcal{L}^{-1}\big\{\frac{1}{s^n}\big\}=\frac{1}{(n-1)!}t^{n-1}$,
$\mathcal{L}^{-1}\big\{\frac{\omega}{s^2+\omega^2}\big\}=\sin(\omega t)$,
$\mathcal{L}^{-1}\big\{\frac{s}{s^2+\omega^2}\big\}=\cos(\omega t)$.

Theorem.
(1) $\mathcal{L}^{-1}$ is linear;
(2) $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}\mathcal{L}^{-1}\{F(s)\}$;
(3) $\mathcal{L}^{-1}\{F(as)\}=\frac{1}{a}\mathcal{L}^{-1}\{F(s)\}\big|_{t/a}$;
(4) $\mathcal{L}^{-1}\{sF(s)\}=\big[\mathcal{L}^{-1}\{F(s)\}\big]'+\mathcal{L}^{-1}\{F(s)\}(0^+)$;
(5) $\mathcal{L}^{-1}\{F'(s)\}=-t\mathcal{L}^{-1}\{F(s)\}$.

(2) $F(s-a)\mathrel{\widehat{=}}e^{at}f(t)$;
(3) $F(as)\mathrel{\widehat{=}}\frac{1}{a}f\big(\frac{t}{a}\big)$;
(4) $sF(s)\mathrel{\widehat{=}}f'(t)'+f(0^+)$;
(5) $F'(s)\mathrel{\widehat{=}}-tf(t)$.

Fact.
Let f be a function defined at least on an interval $[a,b)$.
Then the function $f(t)[H(t-a)-H(t-b)]$ has values

$$f(t)[H(t-a)-H(t-b)] = \begin{cases} f(t), & t\in[a,b); \\ 0, & \text{elsewhere.} \end{cases}$$

Theorem.
Let $f\in\mathcal{L}_0$, $f \mathrel{\widehat{=}} F$. Then for every $a>0$ we have

$$\begin{aligned} \mathcal{L}\{f(t-a)H(t-a)\} &= e^{-as}\mathcal{L}\{f(t)H(t)\}; \\ \mathcal{L}^{-1}\{e^{-as}F(s)\} &= f(t-a)\cdot H(t-a). \end{aligned}$$

$$\mathcal{L}\{f(t)H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)H(t)\}.$$

Theorem.
Let f be a function that is T-periodic on $[0,\infty)$. Denote one period as $f_T(t)=f(t)[H(t)-H(t-T)]$. Then $\mathcal{L}\{f(t)\} = \dfrac{\mathcal{L}\{f_T(t)\}}{1-e^{-sT}}$.

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha} \qquad \mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f\}(s-a)$$
$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{t\,f(t)\} = -[\mathcal{L}\{f\}(s)]'$$
$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f\}(s) - f(0^+)$$
$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f\}(s) - sf(0^+) - f'(0^+)$$
$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\Big\{\int_0^t f(u)\,du\Big\} = \frac{1}{s}\mathcal{L}\{f\}(s)$$
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{f(t)\cdot H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)H(t)\}$$
$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}\cdot H(t-a)$$

Assume: $f(t) \mathrel{\widehat{=}} F(s)$

$$e^{\alpha t} \mathrel{\widehat{=}} \frac{1}{s-\alpha} \qquad e^{at}f(t) \mathrel{\widehat{=}} F(s-a)$$
$$\sin(\omega t) \mathrel{\widehat{=}} \frac{\omega}{s^2+\omega^2} \qquad t\,f(t) \mathrel{\widehat{=}} -F'(s)$$
$$\cos(\omega t) \mathrel{\widehat{=}} \frac{s}{s^2+\omega^2} \qquad f'(t) \mathrel{\widehat{=}} sF(s) - f(0^+)$$
$$1 \mathrel{\widehat{=}} \frac{1}{s} \qquad f''(t) \mathrel{\widehat{=}} s^2F(s) - sf(0^+) - f'(0^+)$$
$$t \mathrel{\widehat{=}} \frac{1}{s^2} \qquad \int_0^t f(u)\,du \mathrel{\widehat{=}} \frac{1}{s}F(s)$$
$$t^n \mathrel{\widehat{=}} \frac{n!}{s^{n+1}} \qquad f(t-a)\cdot H(t-a) \mathrel{\widehat{=}} e^{-as}F(s)$$
$$\mathcal{L}\{f(t)\cdot H(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)H(t)\}$$
$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\cdot H(t-a)$$