DEN: Errors in calculations

Definition.
Let x be a number and $\hat{x}$ its estimate. Then we define

absolute error $E_x = x - \hat{x}$

and **relative error** $\varepsilon_x = \frac{|E_x|}{|x|}$ if $x \neq 0$.

By an **error estimate** we mean any number e_x satisfying $|E_x| \leq e_x$.

Definition.
By a **floating point representation** of a number x
with respect to base β, with precision of p significant digits
we mean the best approximation $fl(x)$ of x that can be written as

$$fl(x) = d_1.d_2d_3 \cdots d_p \times \beta^e,$$

where $d_1 \in \{1, \dots, \beta - 1\}$ and $d_2, \dots, d_p \in \{0, 1, \dots, \beta - 1\}$.
The number e is called the exponent, the part $d_1.d_2 \cdots d_p$ is called the significand or the mantissa.

Fact.
Assume that a number x was represented as $\hat{x}$ in floating point representation with base β and precision p. Then the relative error is bounded as follows:

$$\varepsilon_x \leq \frac{1}{2}\beta \cdot \beta^{-p}$$

Fact.
Consider real numbers x, y and their estimates $\hat{x}, \hat{y}$. Then the following are true:

$|E_{x+y}| \leq |E_x| + |E_y|$ $\qquad$ $\varepsilon_{x+y} \leq \max(\varepsilon_x, \varepsilon_y)$ for $x, y > 0$;

$|E_{x-y}| \leq |E_x| + |E_y|$ $\qquad$ $\varepsilon_{x-y} \leq \frac{|x|+|y|}{|x-y|} \max(\varepsilon_x, \varepsilon_y)$;

$|E_{x\cdot y}| \leq |y| \cdot |E_x| + |\hat{x}| \cdot |E_y|$ $\qquad$ $\varepsilon_{x\cdot y} \leq \varepsilon_x + (1 + \varepsilon_x)\varepsilon_y$;

$|E_{x/y}| \leq \frac{1}{|y|}\big(|E_x| + |E_y|\frac{|\hat{x}|}{|\hat{y}|}\big)$ $\qquad$ $\varepsilon_{x/y} \leq \varepsilon_x + \varepsilon_y \frac{1+\varepsilon_x}{1-\varepsilon_y}$ for $\varepsilon_y < 1$;

$|E_{1/x}| \leq \frac{|x|}{|\hat{x}|}|E_x|$ $\qquad$ $\varepsilon_{1/x} \leq \frac{1}{1-\varepsilon_x}\varepsilon_x$ for $\varepsilon_x < 1$.

Fact.
Assume that there is a rounding error $\varepsilon > 0$ on input, then for $x, y > 0$ we (almost) have

$$\begin{aligned}
\varepsilon_{ax+y} &\leq \varepsilon, \\
\varepsilon_{x-y} &\leq \frac{x+y}{|x-y|}\varepsilon, \\
\varepsilon_{x\cdot y} &\leq 2\varepsilon, \\
\varepsilon_{x/y} &\leq 2\varepsilon, \\
\varepsilon_{1/x} &= \varepsilon.
\end{aligned}$$