

DEN: Typical midterm test

1. Find the solution of the IVP

$$y' = -3\frac{y-3}{x}, \quad y(-1) = 5.$$

2. a) Find a general solution of the equation

$$y'' + 2y' + 5y = 0.$$

- b) Discuss the behaviour of a typical solution at infinity.
c) Find the solution given by initial conditions $y(0) = 0$, $y'(0) = 2$.

3. Consider the equation $y'' - 2y' = 13e^{3x} + 23$.
Estimate the general form of its particular solution y_p .

4. Find a general solution of the system

$$\begin{aligned}y_1' &= 7y_1 - 6y_2 \\y_2' &= 6y_1 - 6y_2.\end{aligned}$$

Solutions

1. Equation demands that $x \neq 0$.

Separation: $\int \frac{dy}{y-3} = -3 \int \frac{dx}{x}$. Stat. sol. $y(x) = 3$.

Integration: $\ln|y-3| = -3 \ln|x| + c = \ln|\frac{1}{x^3}| + c$, trick with $C = \pm e^c \neq 0$, hence $y(x) = \frac{C}{x^3} + 3$.

Existence: $x \neq 0$. From the procedure $y \neq 3$, this is true for $C \neq 0$. The choice $C = 0$ includes the stat. sol.

Hence general solution $y(x) = \frac{C}{x^3} + 3$, $x \neq 0$.

Initial cond.: $\frac{C}{(-1)^3} + 3 = 5$ yields $C = -2$. We want interval with $x_0 = -1$, hence solution $y(x) = 3 - \frac{2}{x^3}$, $x \in (-\infty, 0)$.

2. a) $\lambda^2 + 2\lambda + 5 = 0 \implies \lambda = -1 \pm 2i$.

$y(x) = a e^{-x} \sin(2x) + b e^{-x} \cos(2x)$, $x \in \mathbb{R}$.

b) For $x \sim \infty$ we have $y(x) \rightarrow 0$.

c) $y'(x) = -a e^{-x} \sin(2x) + 2a e^{-x} \cos(2x) - b e^{-x} \cos(2x) - 2b e^{-x} \sin(2x)$.

I.C.:

$$0 + b = 0$$

$$\implies a = 1, b = 0.$$

$$-0 + 2a - b - 0 = 2$$

Solution: $y(x) = e^{-x} \sin(2x)$, $x \in \mathbb{R}$.

3. Two different types on the right, an exponential with $\alpha = 3$ and a polynomial. First guess is therefore $A e^{3x} + B$.

Corrections? The left-hand side (homogeneous equation) has characteristic numbers

$$\lambda^2 - 2\lambda = 0 \implies \lambda = 0, 2.$$

Right-hand side: The exponential part is described by the parameter $\lambda = 3$, no correction. The polynomial part does not feature any exponentials nor sines/cosines, hence it is described by the parameter $\lambda = 0$. This matches one of the characteristic numbers (multiplicity one), hence there will be (a single) correction.

Conclusion: The estimated form is $y_p = A e^{3x} + Bx$.

4. We work with the matrix

$$\begin{pmatrix} 7 & -6 \\ 6 & -6 \end{pmatrix}.$$

We find eigenvalues:

$$\det \begin{pmatrix} 7 - \lambda & -6 \\ 6 & -6 - \lambda \end{pmatrix} = -(7 - \lambda)(6 + \lambda) + 36 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.$$

We found $\lambda = -2, 3$. We find eigenvectors and solutions for the fundamental system:

$$\lambda = -2: \begin{pmatrix} 9 & -6 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 3v_1 - 2v_2 = 0, \text{ choose } v_2 = 3, \text{ then } v_1 = 2, \vec{y}_a(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2x}.$$

$$\lambda = 3: \begin{pmatrix} 4 & -6 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2v_1 - 3v_2 = 0, \text{ choose } v_2 = 2, \text{ then } v_1 = 3, \vec{y}_b(x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3x}.$$

General solution $\vec{y}(x) = a \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2x} + b \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3x}$, that is,

$$y_1(x) = 2a e^{-2x} + 3b e^{3x},$$

$$y_2(x) = 3a e^{-2x} + 2b e^{3x}, x \in \mathbb{R}.$$