

$$y''' + 4y'' + 4y' = 0$$

$$\rightarrow \lambda^3 + 4\lambda^2 + 4\lambda = 0$$

$$\lambda(\lambda^2 + 4\lambda + 4) = 0$$

$$\lambda(\lambda + 2)^2 = 0$$

→ general solution  
→ asymptotic growth  
at  $\infty$

$$\rightarrow \lambda = 0, \lambda = -2 \text{ (2x)}$$

$$\text{f.s. } \left\{ \underbrace{e^{0 \cdot x}}_1, e^{-2 \cdot x}, x \cdot e^{-2 \cdot x} \right\}$$

general sol:  $y(x) = a \cdot 1 + b \cdot e^{-2x} + c \cdot x e^{-2x}, x \in \mathbb{R}$

→ as  $x \rightarrow \infty$ :  $y(x) \rightarrow a$

$$y(x) \sim a$$

$$y^{(5)} - 4y''' = 0$$

$$y^{(5)} - 4y''' = 0$$

$$\rightarrow \lambda^5 - 4\lambda^3 = 0$$

$$\lambda^3(\lambda^2 - 4) = 0$$

$$\lambda^3(\lambda - 2)(\lambda + 2) = 0$$

→ general sol.

→ asymptotic rate of growth at  $\infty$  of a typical solution

$$\lambda = 0 (3x), \pm 2$$

$$\text{f.s. } \left\{ \frac{e^{0 \cdot x}}{1}, x \cdot \frac{e^{0 \cdot x}}{1}, x^2 \cdot \frac{e^{0 \cdot x}}{1}, e^{2x}, e^{-2x} \right\}$$

$$\text{general sol: } y(x) = a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot e^{2x} + f \cdot e^{-2x}, x \in \mathbb{R}$$

→ as  $x \sim \infty$ ,

$$y(x) \sim d \cdot e^{2x}$$

if  $d \neq 0$

if  $d = 0$ :  $y(x) \sim c \cdot x^2$   
if  $c \neq 0$

$$y'''' + 4y'' = 0$$

$$\rightarrow \lambda^5 + 4\lambda^3 = 0$$

$$\lambda^3(\lambda^2 + 4) = 0$$

→ gener. sol.

→ a.v.g. of t.s. at  $\infty$

$$\rightarrow \lambda = 0 \text{ (3x)}, \lambda = \pm 2i$$
$$= \underline{0} \pm \underline{2i}$$

f.s.  $\{e^{0 \cdot x}, x \cdot e^{0 \cdot x}, x^2 \cdot e^{0 \cdot x}, \underline{e^{0 \cdot x} \cdot \cos(2x)}, \underline{e^{0 \cdot x} \cdot \sin(2x)}\}$

general sol:  $y(x) = a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot \cos(2x) + f \cdot \sin(2x)$ ,  
 $x \in \mathbb{R}$

$$\lambda = 0 + 2i \rightarrow e^{(0+2i)x} = e^{0 \cdot x} e^{2i \cdot x} = e^{0 \cdot x} (\cos(2x) + i \sin(2x))$$
$$= \underbrace{1 \cdot \cos(2x)}_{\rightarrow \text{Re}} + i \underbrace{1 \cdot \sin(2x)}_{\rightarrow \text{Im}} \quad \left. \vphantom{e^{(0+2i)x}} \right\}$$

→ as. growth:  $0 \rightarrow a$   $0 \rightarrow \infty$   $0 \rightarrow \infty$   $0 \rightarrow \text{osc}$   $0 \rightarrow \text{osc}$

$$y(x) \sim C \cdot x^2 \text{ as } x \rightarrow \infty.$$

$$y'' - 4y' + 13 = 0$$

$$\rightarrow \lambda^2 - 4\lambda + 13 = 0$$

$$(\lambda - 1)(\lambda - 13) = 0$$

$\rightarrow$  g.s.

$\rightarrow$  a.s.o.t.s.a.i.

$$\underline{\underline{\lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm \frac{1}{2}\sqrt{-36}}}$$

$$= 2 \pm \sqrt{-9} = \underline{\underline{2 \pm 3i}}$$

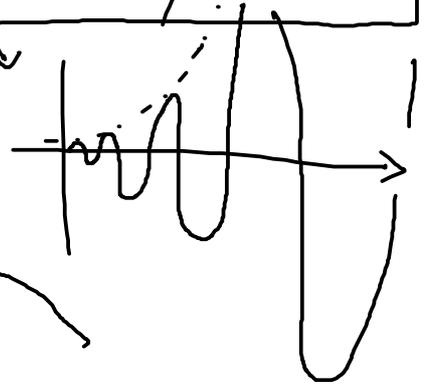
general sol.

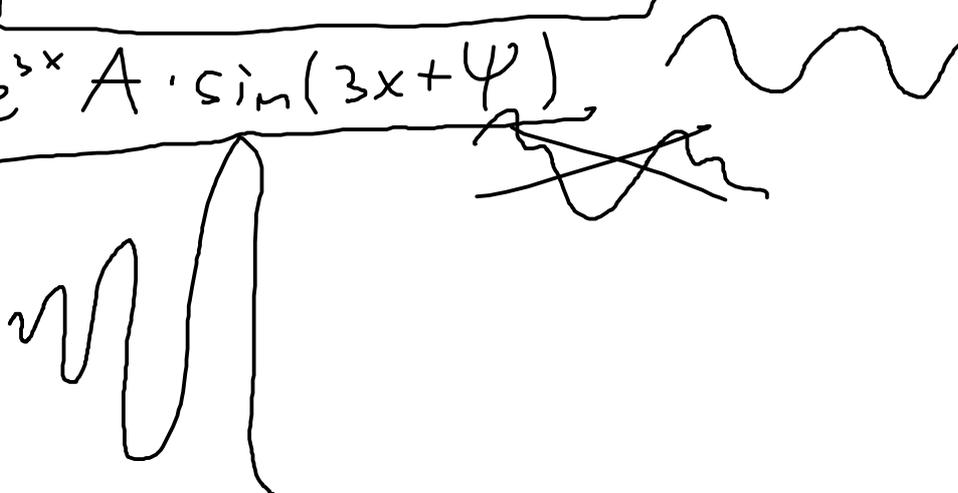
$$y(x) = a \cdot e^{2x} \cos(3x) + b \cdot e^{2x} \sin(3x), \quad x \in \mathbb{R}$$

$\downarrow$   $\omega$

$\downarrow$

$\rightarrow$  no answer



$$y(x) = e^{3x} \left[ a \cdot \cos(3x) + b \cdot \sin(3x) \right] \quad \left. \begin{array}{l} A = \sqrt{a^2 + b^2} \\ \varphi \end{array} \right\}$$
$$= e^{3x} \cdot A \cdot \cos(3x + \varphi)$$
$$= e^{3x} A \cdot \sin(3x + \varphi)$$


$$y'' - y' - 6y = 0$$

solution satisfying  $y(0) = -4$   
 $y'(0) = 13$

→ general sol.

$$\lambda^2 - \lambda - 6 = 0$$
$$(\lambda + 2)(\lambda - 3) = 0$$

$$\lambda = -2, 3$$

gen-sol.

$$a \cdot e^{3x} + b \cdot e^{-2x} \quad \checkmark$$

$$y(x) = a \cdot e^{-2x} + b \cdot e^{3x}, x \in \mathbb{R}$$

$$y' = -2a e^{-2x} + 3b e^{3x}$$

→ want:

$$\begin{cases} y(0) = a \cdot e^{-2 \cdot 0} + b e^{3 \cdot 0} \stackrel{?}{=} -4 \\ y'(0) = -2a \cdot e^{-2 \cdot 0} + 3b e^{3 \cdot 0} \stackrel{?}{=} 13 \end{cases} \Rightarrow$$

$$\begin{cases} a + b = -4 \\ -2a + 3b = 13 \end{cases} \Rightarrow$$

$$\Rightarrow 2 \times (\#1) + (\#2): 5b = 5 \rightarrow \underline{b = 1} \quad \underline{a = -5} \quad \checkmark$$

solution:  $\underline{y(x) = -5e^{-2x} + e^{3x} = e^{3x} - 5e^{-2x}, x \in \mathbb{R}}$

a) Find some initial conditions at  $x_0 = 0$  so that the resulting solution satisfies  $y(1) = e^3$ ,  $y'(1) = 3e^3$ .

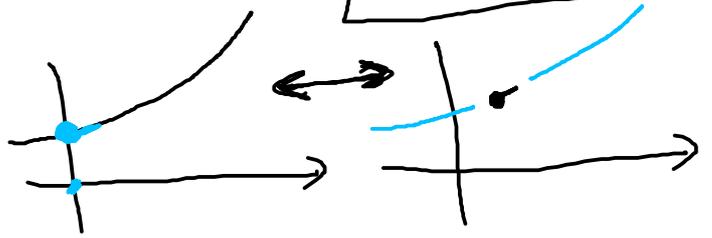
general sol:  $y(x) = ae^{-2x} + be^{3x}$ ,  $x \in \mathbb{R}$

$$\begin{aligned} \rightarrow y(1) &= ae^{-2} + be^3 \stackrel{?}{=} e^3 \\ y'(1) &= -2ae^{-2} + 3be^3 \stackrel{?}{=} 3e^3 \end{aligned}$$

$$\cancel{be^3} = \cancel{be^3} \quad \begin{cases} b=1 \\ a=0 \end{cases}$$

I want  $y(x) = e^{3x}$ ,  $x \in \mathbb{R}$

$$\begin{cases} y(0) = 1 \\ y'(0) = 3 \end{cases}$$



b) Find some init. cond. at  $x_0 = 0$  so that the corresp. solution is bounded on  $(0, \infty)$ .

→ I want  $b=0$ . Say,  $y(x) = 3e^{-2x}$   $y(x) = ae^{-2x} + be^{3x}$   
→  $y' = -2ae^{-2x}$

$$\begin{cases} y(0) = 3 \\ y'(0) = -26 \end{cases}$$

$$\begin{cases} y(x) = 0 \leftarrow a=0, b=0 \\ (y(0) = 0, y'(0) = 0) \end{cases}$$

↖ non-trivial solution      ↖ trivial sol.

→ work or not?

c) Develop a test that recognizes which I.C. at  $x_0=0$  lead to a solution bounded on  $(0, \infty)$ .

→ I want  $y(x) = ae^{-2x}$

I.C.:  $\begin{cases} y(0) = a \\ y'(0) = -2a \end{cases}$

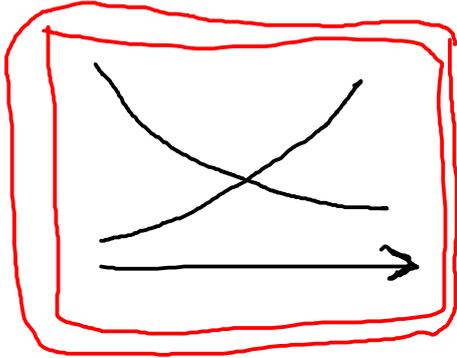
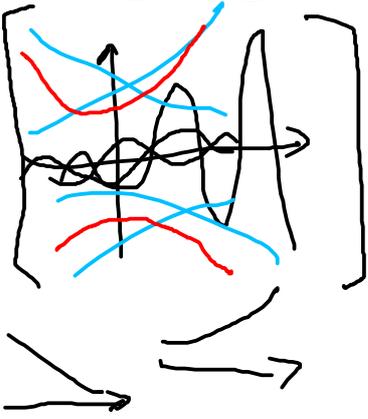
$y'(0) = -2y(0)$

$\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$

$B \stackrel{?}{=} -2A \rightarrow \text{true} \Rightarrow$  bdd sol. on  $(0, \infty)$   
 $\downarrow$  not true  $\rightarrow \begin{cases} \infty \\ -\infty \end{cases}$

Given  $y'' - by' + apy = 0$   $p \in \mathbb{R}$  ... parameter

For which values of  $p$  it will be guaranteed that the solution set does not include both a positive increasing sol. and a positive decreasing sol.



to prevent

$ae^{\lambda x} \quad a, \lambda > 0$

$ae^{\lambda x} \quad a > 0, \lambda < 0$

$$y'' - 6y' + 9p y = 0$$

$$\rightarrow \lambda^2 - 6\lambda + 9p = 0$$

$$\lambda = \frac{6 \pm \sqrt{36 - 36p}}{2}$$

$$= 3 \pm \sqrt{9 - 9p} = \underline{3 \pm 3\sqrt{1-p}}$$

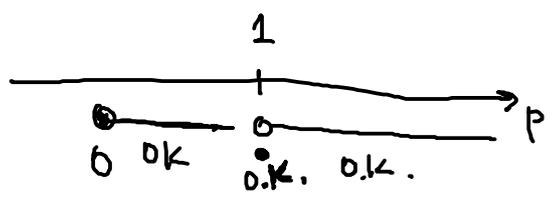
cases  
 $\xrightarrow{1 < p}$

$$1 - p < 0 \rightsquigarrow 3 \pm \sqrt{|1-p|} \cdot i$$

$$y(x) = a \cdot e^{3x} \cos(\omega x) + b \cdot e^{3x} \sin(\omega x)$$

if  $a = b = 0$   $y(x) = 0$   
 not incr. / decr.  
 not pos.

  
 it is ok.



}  $p \geq 0$  is o.k.

$\rightarrow 1-p=0 \rightarrow \lambda=3 \quad (2x)$

$p=1$

$y(x) = ae^{3x} + bx e^{3x}$

if  $a, b > 0 \Rightarrow$  increasing positive

decreasing, positive?  $b > 0 \rightarrow$  not decreasing



$p=1$  is fine.

$p < 1$

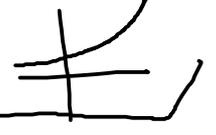
$\rightarrow 1-p > 0 \rightarrow \lambda = 3 \pm 3\sqrt{1-p}$

$3 - 3\sqrt{1-p}$

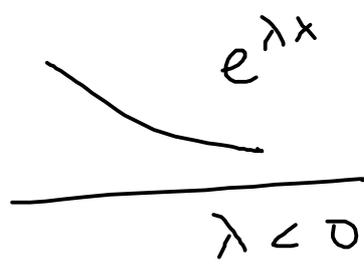
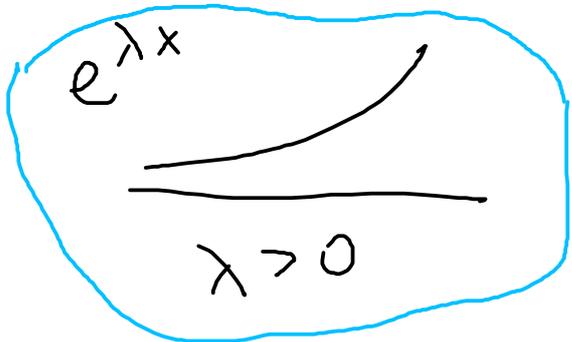
if  $< 0$  then ~~prevent this~~

$3 + 3\sqrt{1-p} > 0$

$a \cdot e^{(3+\sqrt{\quad})x}$   
 $a > 0$

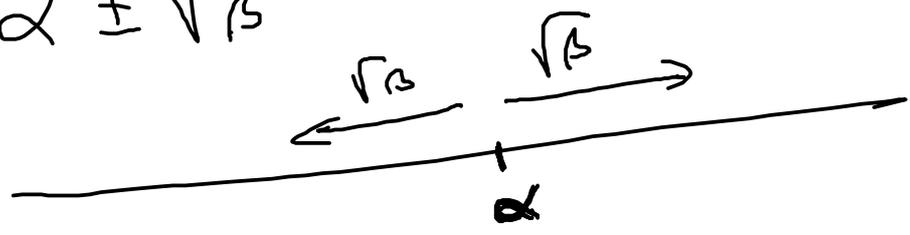


We want  $3 - 3\sqrt{1-p} \geq 0 \Leftrightarrow 3 \geq 3\sqrt{1-p} \Leftrightarrow 1 \geq \sqrt{1-p} \Leftrightarrow$   
 $\Leftrightarrow 1 \geq 1-p \Leftrightarrow p \geq 0$



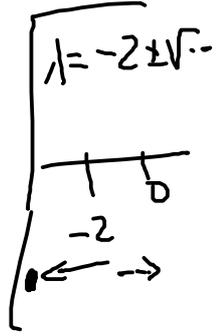
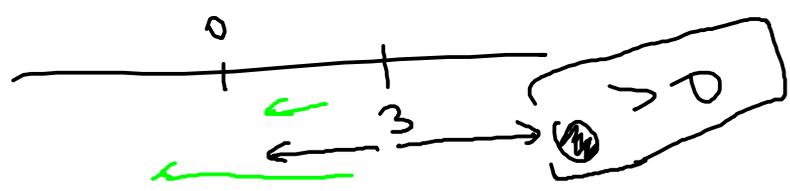
$$\lambda = \alpha \pm \sqrt{\beta}$$

$$\beta > 0$$



$$\lambda = \underline{3} \pm 3\sqrt{1-p}$$

$\alpha$



→ linear <sup>diff</sup> equation

• →  $y'' + ay' + by = 0$   
a, b, c, ... constant

$\lambda = \dots \rightarrow y(x) = \dots$

• →  $y' + a(x)y = b(x)$   
↳  $y' + a(x)y = 0$   
(by sep.)

do not need  $a(x) = \text{const.}$

if  $a(x) \text{ const}$

↓  
λ

linear diff. eq.

const. coeff. (1) order 1

$$y' - 2y = \frac{1}{x} e^{2x}, \quad y(-1) = 0$$

$x \neq 0$

→ general sol. of given eq.  
 → hom. version:  $y' - 2y = 0 \Rightarrow \lambda - 2 = 0$   
 $\lambda = 2$   
 $y_h(x) = a e^{2x}$

→ variation:  $y_p(x) = a(x) e^{2x}$

$[a'(x) \cdot e^{2x} + a(x) \cdot 2 \cdot e^{2x}] - 2 \cdot a(x) e^{2x} \stackrel{!}{=} \frac{1}{x} e^{2x} \Rightarrow a'(x) = \frac{1}{x}$

$\Rightarrow a(x) = \ln|x|$ , so  $y_p(x) = \ln|x| \cdot e^{2x}$ , trick  $y_p + y_h$

$\Rightarrow$  general sol.  $y(x) = \ln|x| e^{2x} + a e^{2x}, x \neq 0$

→ init. cond.:  $0 + a \cdot e^{-2} = 0 \Rightarrow a = 0$

$y(x) = \ln|x| \cdot e^{2x}, x \in (-\infty, 0)$