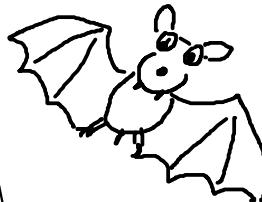


$$y' + y = 4e^x + x, \quad y(1) = 2e - \frac{2}{e}$$



1. general sol.
 a) homog. version y_h
 b) R.H.S. → guessing
 (variation) → y_p
 2. init. cond.

1. gensol. a) hom: $y' + y = 0 \Rightarrow \lambda + 1 = 0 \rightarrow \lambda = -1$, $y_h(x) = ae^{-x}$

b) $b(x) = 4e^x + x$
 $\rightarrow 4e^x \xrightarrow{\text{base}} A e^x \xrightarrow[\lambda=1]{\lambda=1+0i} \text{no correction} \rightarrow Ae^x$
 $\rightarrow x \xrightarrow{\text{base}} Bx + C \xrightarrow[\text{hom matr.}]{\lambda=0+0i=0} \text{no correction} \rightarrow Bx + C$

guess: $y_p(x) = Ae^x + Bx + C$.
 $\boxed{A=0 \rightarrow ae^{0 \cdot x} = a}$

$$y_1 + y_2 = 4e^x + x$$

$$y_p(x) = Ae^x + Bx + C$$

$$\mathcal{L}[y_p] = [Ae^x + Bx + C]' + [Ae^x + Bx + C]$$

$$= \underline{Ae^x} + \underline{B} + \underline{Ae^x} + \underline{Bx} + \underline{C} = \underline{2Ae^x} + \underline{Bx} + \underline{(B+C)}$$

$$\begin{cases} 2A = 4 \\ B = 1 \\ B+C = 0 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 1 \\ C = -1 \end{cases}$$

$$y_p(x) = 2e^x + x - 1$$

General sol: $y(x) = \underline{2e^x + x - 1} + a e^{-x}, x \in \mathbb{R}$

$$2. y(1) = 2e - \frac{2}{e} \quad | \quad 2e + 1 + a e^{-1} \stackrel{?}{=} 2e - \frac{2}{e}$$

sol:

$$y(x) = 2e^x + x - 1 + 2e^{-x}, x \in \mathbb{R}$$

$$\frac{a}{e} \stackrel{?}{=} \frac{2}{e} \rightarrow a = 2$$

$$5Axe^x + \beta x + (\beta + c) = 4e^x + x + 0$$

$y''' - 4y' =$	$\lambda = 0, -2, 2$	$y'' - 4y' + 4y =$	$\lambda = 2 \text{ (2x)}$	$L\{y\} =$
$x \cdot (Ax + B) \cdot e^{-2x}$		$(Ax + B) \cdot e^{-2x}$		$= x \cdot e^{-2x}$
$A e^{2x} \cos(x) + B e^{2x} \sin(x)$	no corr.	$A e^{2x} \cos(x) + B e^{2x} \sin(x)$	no corr.	$= 3e^{2x} \cos(x)$
$(Ax^2 + Bx + C)x$		$Ax^2 + Bx + C$	no corr.	$= 5x^2 - 1$
$(Ax + B) \cos(\pi x) + (Cx + D) \sin(\pi x)$	no corr.	$(Ax + B) \cos(\pi x) + (Cx + D) \sin(\pi x)$	no corr.	$= \frac{Bx}{\lambda} \cos(\pi x)$ $\lambda = \pi i$
$A e^x + x \cdot B e^{2x}$		$A e^x + x^2 \cdot B e^{2x}$		$= e^x - 3e^{2x}$
$x \cdot (Ax + B) + C \sin(3x) + D \cos(3x)$	no corr.	$(Ax + B) + C \cdot \sin(3x) + D \cos(3x)$		$= 2x - \frac{\sin(3x)}{\lambda}$ $\lambda = 0 \quad \quad \lambda = 3i$

Find the solution of $y'' - y = 4 \sin(x) - 2x$
 satisfying $y(0) = 3, y'(0) = 0$

1. gensol. a) hom: $\lambda^2 - \lambda = 0 \Leftrightarrow \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0, 1$

$$y_h(x) = a e^{0x} + b e^x \quad [y_h = a + b e^x]$$

b) $b(x) = 4 \sin(x) - 2x$

$$\rightarrow 4 \sin(x) \rightarrow A \cdot \sin(x) + B \cdot \cos(x) \xrightarrow[\lambda=i]{\lambda=0+i \cdot i} \text{no corr.}$$

$$\rightarrow -2x \rightarrow Cx + D \xrightarrow[\text{match!}]{\lambda=0} (Cx+D) \cdot x$$

guess $y_p(x) = A \sin(x) + B \cos(x) + Cx^2 + Dx$

$$\begin{aligned} L[y_p] &= \left[-A \sin(x) - B \cos(x) + 2C \right] - \left[A \cos(x) - B \sin(x) + 2Cx + D \right] \\ &= \underline{-A + B} \sin(x) + \underline{-A - B} \cos(x) - \underline{2Cx} + \underline{(2C - D)} \\ &\stackrel{?}{=} \underline{4} \sin(x) + \underline{0} \cos(x) - \underline{2x} + \underline{0} \end{aligned}$$

$$\left. \begin{array}{l} -A + B = 4 \\ -A - B = 0 \\ -2C = -2 \\ 2C - D = 0 \end{array} \right\} \oplus -2A = 4 \rightarrow A = -2 \quad \left. \begin{array}{l} B = 2 \\ C = 1 \\ D = 2 \end{array} \right\} \quad \begin{array}{l} y_p = -2 \sin(x) + 2 \cos(x) \\ \cdot \quad + x^2 + 2x \end{array}$$

general sol

$$y(x) = 2 \cos(x) - 2 \sin(x) + x^2 + 2x + a + b e^x, x \in \mathbb{R}$$

$$\begin{array}{l} y = 2\cos(x) - 2\sin(x) + x^2 + 2x + a + be^x \\ y' = -2\sin(x) - 2\cos(x) + 2x + 2 + be^x \end{array} \quad \left| \begin{array}{l} y(0) = 3 \\ y'(0) = 0 \end{array} \right.$$

$$\begin{bmatrix} 2 \cdot 1 - 2 \cdot 0 + 0^2 + 0 + a + be^0 & \stackrel{?}{=} 3 \\ -2 \cdot 0 - 2 \cdot 1 + 0 + 2 + be^0 & \stackrel{?}{=} 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a + b = 1 \\ b = 0 \end{bmatrix} \rightarrow a = 1.$$

Sol: $y(x) = 2\cos(x) - 2\sin(x) + x^2 + 2x + 1, x \in \mathbb{R}$

$$y''' - y'' + y' - y = 10e^{2x} - x \quad \begin{cases} \text{behaviour of} \\ \text{a typical solution at } \infty \end{cases}$$

1. a) $\lambda^3 - \lambda^2 + \lambda - 1 = 0 \rightarrow \lambda = 1 \quad (\lambda=1) \cdot (\lambda^2 + 1) = 0 \rightarrow \lambda = \pm i$

$$y_h = a e^x + b \cos(x) + c \sin(x)$$

b) $\rightarrow 10e^{2x} \rightarrow A e^{2x} \xrightarrow[\text{no corr}]{\lambda=2} A e^{2x} \quad \left. \begin{array}{l} \\ \end{array} \right\} y_p = A e^{2x} + Bx + C$
 $\rightarrow -x \rightarrow Bx + C \xrightarrow[\text{no corr}]{\lambda=0} Bx + C \quad \left. \begin{array}{l} \\ \end{array} \right\} y_p = A e^{2x} + Bx + C$

$$\begin{aligned} L[y_p] &= \left[8Ae^{2x} \right] - \left[4Ae^{2x} \right] + \left[2Ae^{2x} + B \right] - \left[Ae^{2x} + Bx + C \right] \\ &= 5Ae^{2x} - Bx + (B - C) \xrightarrow{\substack{? \\ 10}} 10e^{2x} - x + 0 \quad \left. \begin{array}{l} A=2 \\ B=1 \\ C=1 \end{array} \right\} \end{aligned}$$

General sol: $y(x) = 2e^{2x} + x + 1 + ae^x + b \cos(x) + c \sin(x), x \in \mathbb{R}$

$\lim_{x \rightarrow \infty} y(x) \sim 2e^{2x}$ at ∞

$$\rightarrow y'' - y = 9e^{2x} \rightarrow \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1 \quad (A=3)$$

$y(x) = Ae^{2x} + ae^x + be^{-x} \sim Ae^{2x}$ at ∞ .

Find the behavior of typical solutions at infinity

$\hookrightarrow y_h$

\hookrightarrow guess y_p

answer

$$\rightarrow y'' + 3y' + 2y = \sin(x) + 3\cos(x) \quad (A=1, B=0)$$

$$\rightarrow \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) = 0 \Rightarrow \lambda = -1, -2$$

$y(x) = A\sin(x) + B\cos(x) + ae^{-x} + be^{-2x}$ no dominant terms at ∞

$$\rightarrow y'' - 4y' + 4y = e^{2x} \rightarrow \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0 \Rightarrow \lambda = 2$$

$$(A=\frac{1}{2}, B=0)$$

$y(x) = x^2 Ae^{2x} + ae^{2x} + bx^2 e^{2x} \sim Ax^2 e^{2x}$ at ∞

$$\rightarrow y'' - 4y = -8x \rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

$y(x) = Ax + B + ae^{2x} + be^{-2x} \sim ae^{2x}$ at ∞ .

$(A=2, B=0)$

$A \sin(x+4)$
+ ...
 $\approx A \sin(x+4)$

$$y'' - 3y' + 2y = e^{4x} + 13$$

LC  $\left[A e^{4x} + B \right] = y_p$

$$\left[A e^{4x} \right] = \dots = e^{4x}$$

$$\left[B \right] = \dots = 13$$

$$A =$$

$$B =$$

$$y'' + 2y' = 6e^x - 2e^{-2x} + 8\cos(2x) + 4x \quad \begin{cases} y(0) = 2 \\ y'(0) = 2 \end{cases}$$

a) $\lambda^2 + 2\lambda = 0 \rightarrow \lambda = 0, -2$

b) $y_p(x) = Ae^x + Bx \cdot e^{-2x} + (C\cos(2x) + D\sin(2x) + Ex + F)$

$\underbrace{\lambda = 1}_{\lambda = -2} \quad \underbrace{\lambda = 2i}_{\lambda = 0}$

$$\begin{aligned} L[Ae^x + Bx \cdot e^{-2x}] &= [Ae^x - 2Be^{-2x} - 2B'e^{-2x} + 4Be^{-2x}] + 2[Ae^x + Be^{-2x} - 2Bx \cdot e^{-2x}] = \\ &= 3Ae^x - 2Be^{-2x} \stackrel{?}{=} 6e^x - 2e^{-2x} \rightarrow \begin{cases} A = 2 \\ B = 1 \end{cases} \\ L[C\cos(2x) + D\sin(2x) + Ex^2 + Fx] &= [-4C\cos(2x) - 4D\sin(2x) + 2E] + \\ + 2[-2C\sin(2x) + 2D\cos(2x) + 2Ex + F] &= \underbrace{(-4C + 4D)\cos(2x)}_8 + \underbrace{(4C - 4D)\sin(2x)}_0 + \underbrace{4Ex + (2E + 2F)}_{E=1, F=-1} \end{aligned}$$

$$Ae^x + Bx \cdot e^{-2x} + C\cos(2x) + D\sin(2x) + E(x+F)$$

general sol: $y(x) = \cancel{2e^x} + \cancel{x e^{-2x}} - \cancel{\cos(2x)} + \cancel{\sin(2x)} + x^2 - x + a + b e^{-2x}, x \in \mathbb{R}$

2. i. c.

$$\begin{cases} y(0) = \cancel{2} + 0 - \cancel{1} + 0 + 0 - 0 + a + b = 2 \\ y'(0) = \cancel{2} + \cancel{1} - \cancel{0} + 0 + \cancel{2} + 0 - \cancel{1} - 2b = \cancel{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b = 1 \\ -2b = -2 \end{cases} \rightarrow \begin{array}{l} a = 0 \\ b = 1 \end{array}$$

$$y(x) = 2e^x + x e^{-2x} - \cos(2x) + \sin(2x) + x^2 - x + e^{-2x}, x \in \mathbb{R}$$