

$$\begin{cases} y_1' = 2y_1 - y_2 \\ y_2' = 4y_1 - 3y_2 \end{cases}$$

$$\begin{cases} y_1(0) = 0 \\ y_2(0) = -3 \end{cases}$$

1) general sol.

• elimination from (#2):

$$y_2 = 2y_1 - y_1' \quad \text{(*)}$$

↳ into (#2)

$$2y_1' - y_1'' = 4y_1 - 3(2y_1 - y_1')$$

$$\rightarrow y_1'' + y_1' - 2y_1 = 0$$

$$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$$

→ $\lambda = 1, \lambda = -2$

$$y_1(x) = ae^x + be^{-2x} //$$

$$\text{(*)}: y_2(x) = 2ae^x + 2be^{-2x} - ae^x + 2be^{-2x}$$

$$y_2(x) = ae^x + 4be^{-2x} // \quad x \in \mathbb{R}$$

• matrix appr: $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$

eigenvalues: $\det \begin{pmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{pmatrix} = (\lambda-2) \cdot (\lambda+3) + 4 =$
 $= \lambda^2 + \lambda - 2 \stackrel{!}{=} 0 \Rightarrow \lambda = 1, -2$

eigenvectors

$\lambda = 1$ $\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \rightarrow v_1 - v_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{1 \cdot x} = \vec{y}_a$
 choose $v_2 = 1 \Rightarrow v_1 = 1$

$\lambda = -2$ $\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \rightarrow 4v_1 - v_2 = 0 \Rightarrow \vec{y}_b = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot e^{-2x}$
 choose $v_1 = 1 \Rightarrow v_2 = 4$

general sol: $\vec{y} = a \cdot \vec{y}_a + b \vec{y}_b = a \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x + b \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}, x \in \mathbb{R}$

$\vec{y} = \begin{pmatrix} ae^x + be^{-2x} \\ ae^x + 4be^{-2x} \end{pmatrix} \Rightarrow$ general sol

$y_1(x) = ae^x + be^{-2x}$
 $y_2(x) = ae^x + 4be^{-2x}, x \in \mathbb{R}$

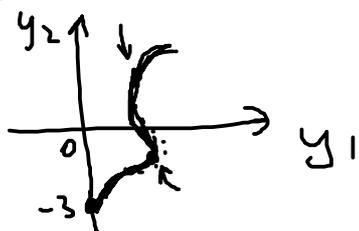
2. I.C. $y_1(0) = 0$ $y_2(0) = -3$ $\left[\begin{array}{l} a \cdot e^0 + b e^0 = 0 \\ a e^0 + 4b e^0 = -3 \end{array} \right]$

$\left[\begin{array}{l} a + b = 0 \\ a + 4b = -3 \end{array} \right]$ $(\#2) - (\#1): 3b = -3 \Rightarrow b = -1$
 $a = 1$

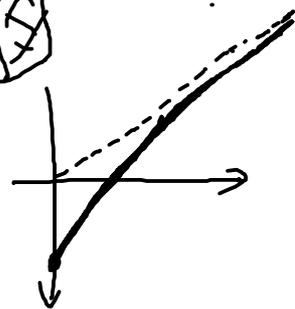
sol:

$y_1(x) = e^x - e^{-2x}$

$y_2(x) = e^x - 4e^{-2x}, x \in \mathbb{R}$



$y_1(0) = 0$
 $y_2(0) = -3$



Bonus:

stat. solutions?

$$y_1' = y_2' = 0$$

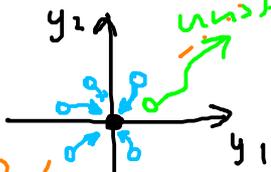
$$\begin{cases} y_1(x) = y_{01} \\ y_2(x) = y_{02} \end{cases} \Big| \vec{y}(x) = \vec{y}_0 \in \mathbb{R}^2$$

$$\begin{cases} 2y_1 - y_2 = 0 \\ 4y_1 - 3y_2 = 0 \end{cases}$$

trivial sol.

$$y_1 = 0 \quad y_2 = 0$$

$$\begin{cases} y_1(x) = 0 \\ y_2(x) = 0, x \in \mathbb{R} \end{cases} \text{ Stat. sol}$$



unstable

$a > 0$

stable
 \forall paths

$(0,0)$
unstable

$$\vec{y}(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a \cdot e^x + \begin{pmatrix} 1 \\ 4 \end{pmatrix} b \cdot e^{-2x}, x \in \mathbb{R}$$

$a = b = 0 \Rightarrow$ stat. sol.

more: $a \neq 0$ or $b \neq 0$. If $a \neq 0$
then $\vec{y}(x) \rightarrow \infty$ in \mathbb{R}^2 as $x \rightarrow \infty$

general sol. of $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -2x_1 + 3x_2 \end{cases} \rightarrow \vec{x}(t)$

matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

eigenvalues: $\det \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (\lambda-1) \cdot (\lambda-3) + 2 = \lambda^2 - 4\lambda + 5 \stackrel{!}{=} 0$

$\lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm \sqrt{-1} = \underline{\underline{2 \pm i}}$

eigenvector

$\lambda = 2+i$ $\begin{pmatrix} -1-i \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \rightarrow -(1+i)v_1 + v_2 = 0$ $\rightarrow \vec{x}_c = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t}$

$v_1 = 1 \Rightarrow v_2 = 1+i$

$\swarrow \searrow$

$\text{Re}(\vec{x}_c) \quad \text{Im}(\vec{x}_c)$

$$\vec{x}_c = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{2t} \cdot e^{it}$$

$$= \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \cdot [\cos(t) + i \sin(t)] \cdot e^{2t}$$

$$= \begin{pmatrix} \cos(t) + i \sin(t) \\ \cos(t) + i \sin(t) + i \cos(t) - \sin(t) \end{pmatrix} e^{2t}$$

$$\lambda = 2+i$$

$$\vec{v} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$\vec{x}_a = \operatorname{Re}(\vec{x}_c) = \begin{pmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^{2t}$$

$$\vec{x}_b = \operatorname{Im}(\vec{x}_c) = \begin{pmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^{2t}$$

$$\vec{x} = a \cdot \vec{x}_a + b \cdot \vec{x}_b$$

$$x_1(t) = a \cos(t) \cdot e^{2t} + b \sin(t) \cdot e^{2t}$$

$$x_2(t) = a(\cos(t) - \sin(t)) \cdot e^{2t} + b(\cos(t) + \sin(t)) \cdot e^{2t}, \quad t \in \mathbb{R}$$

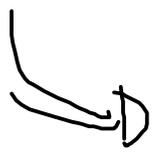
general sol.

$$\bullet \quad \underbrace{y'''' + 3y'' - 2y' - y = 0}_{y_1'''' + 3y_1'' - 2y_1' - y_1 = 0}$$

$$\begin{matrix} y_1 \\ y_2 \\ \vdots \end{matrix}$$

$$y_1' = y_2 \quad \longrightarrow \quad y_2'' + 3y_2' - 2y_2 - y_1 = 0$$

$$y_2' = y_3 \quad \longrightarrow \quad y_3' + 3y_3 - 2y_2 - y_1 = 0 \quad (*)$$



$$(*) \quad \left[\begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 + 2y_2 - 3y_3 \end{array} \right]$$

$$\longrightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

$$\bullet \quad y'''' + 13y'''' - 23y'' + 14y' - 13y = 0 \quad (*)$$

$$\langle y_1'''' + 13y_1'''' - 23y_1'' + 14y_1' - 13y_1 = 0 \rangle$$

$$[y_1' = y_2 \rightarrow y_2'''' + 13y_2'' - 23y_2' + 14y_2 - 13y_1 = 0$$

$$[y_2' = y_3 \rightarrow y_3'' + 13y_3' - 23y_3 + 14y_2 - 13y_1 = 0$$

$$[y_3' = y_4 \rightarrow y_4' + 13y_4 - 23y_3 + 14y_2 - 13y_1 = 0$$

$$\Rightarrow \begin{bmatrix} y_1' = & & & & \\ y_2' = & y_2 & & & \\ y_3' = & & y_3 & & \\ y_4' = & 13y_1 & -14y_2 & +23y_3 & -13y_4 \end{bmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 13 & -14 & 23 & -13 \end{pmatrix} .$$