

$$\begin{bmatrix} x + 2y + 2z = 1 \\ x + y + z = 1 \\ x - y + z = 5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x = 1 - 2y - 2z \\ y = 1 - x - z \\ z = 5 - x + y \end{bmatrix}$$

• Jacobi

$$\begin{bmatrix} x_{k+1} = 1 - 2y_k - 2z_k \\ y_{k+1} = 1 - x_k - z_k \\ z_{k+1} = 5 - x_k + y_k \end{bmatrix}$$

$$\vec{x}_0 = (2, 0, 0)$$

$$\vec{x}_{k+1} = B \cdot \vec{x}_k + \vec{c}$$

$$B_J = \begin{pmatrix} 0 & -2 & -2 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\begin{bmatrix} \text{solution} \\ x = 1 \\ y = -2 \\ z = 2 \end{bmatrix}$$

$$\rho(B_J) < 1 \quad ? \\ > 1 \quad ?$$

	$x$	$y$	$z$
	$x = 1 - 2y - 2z$	$y = 1 - x - z$	$z = 5 - x + y$
$k=0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k=1$	$x_1 = 1$	$y_1 = -1$	$z_1 = 3$
$k=2$	$x_2 = -3$	$y_2 = -3$	$z_2 = 3$
$k=3$	$x_3 = 1$	$y_3 = 1$	$z_3 = 5$
$k=4$	$x_4 = -11$	$y_4 = -5$	$z_4 = 5$
	↓ $(?)$ 1	↓ $(?)$ -2	↓ $(?)$ 2

Guess: diverges (?)

• Gauss-Seidel  $\left[ \begin{array}{l} x_{k+1} = 1 - 2y_k - 2z_k \\ y_{k+1} = 1 - x_{k+1} - z_k \\ z_{k+1} = 5 - x_{k+1} + y_{k+1} \end{array} \right]$   $\boxed{\vec{x}_{k+1} = B \vec{x}_k + \vec{c}}$

$$y_{k+1} = 1 - (1 - 2y_k - 2z_k) - z_k = 2y_k + z_k$$

$$z_{k+1} = 5 - (1 - 2y_k - 2z_k) + (2y_k + z_k) = 4 + 4y_k + 3z_k$$

$$B_{GS} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} \quad \det = -\lambda^3 + 5\lambda^2 - 2\lambda = 0$$

$$\lambda = 0, \frac{1}{2}(5 \pm \sqrt{17})$$

$$\rho(B_{GS}) = \frac{1}{2}(5 + \sqrt{17}) \approx 4.562...$$

$$\rho(B_{GS}) > 1 \Rightarrow GS \text{ diverges.}$$

$$\underline{k=0} \quad x_0 = 2$$

$$y_0 = 0$$

$$z_0 = 0$$

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$$k=1 \quad x_1 = 1$$

$$y_1 = 0$$

$$z_1 = 4$$

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$$k=2 \quad x_2 = -7$$

$$y_2 = 4$$

$$z_2 = 16$$

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$$k=3 \quad x_3 = -39$$

$$y_3 = 24$$

$$z_3 = 68$$

$$\cancel{x} \rightarrow 1$$

$$\cancel{x} \rightarrow -2$$

$$\cancel{x} \rightarrow 2$$

$$\bullet x = 1 - 2y - 2z$$

$$\bullet y = 1 - x - z$$

$$z = 5 - x + y$$

Diverges (?)

$$\begin{cases} 2x + y + z = 2 \\ x + 2y + z = -1 \\ x + y + 4z = 7 \end{cases} \Leftrightarrow$$

sol:  $x=1, y=-2, z=2$

$$\begin{cases} x = 1 - \frac{1}{2}y - \frac{1}{2}z \\ y = -\frac{1}{2} - \frac{1}{2}x - \frac{1}{2}z \\ z = \frac{7}{4} - \frac{1}{4}x - \frac{1}{4}y \end{cases}$$

$$\begin{cases} 2 > 1+1 \\ 2 > 1+1 \\ 4 > 1+1 \end{cases}$$

• Jacobi

Guess:  $\downarrow$  it converges

	(x)	(y)	(z)
	$x = 1 - \frac{1}{2}y - \frac{1}{2}z$	$y = -\frac{1}{2} - \frac{1}{2}x - \frac{1}{2}z$	$z = \frac{7}{4} - \frac{1}{4}x - \frac{1}{4}y$
$k=0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k=1$	$x_1 = 1$	$y_1 = -\frac{3}{2}$	$z_1 = \frac{5}{4}$
$k=2$	$x_2 = \frac{9}{8}$	$y_2 = -\frac{13}{8}$	$z_2 = \frac{15}{8}$
$k=3$	$x_3 = \frac{7}{8} \rightarrow 1 \checkmark$	$y_3 = -2 \rightarrow -2$	$z_3 = \frac{15}{8} \rightarrow 2$

$$\begin{array}{l} k=0 \quad x_0 = 2 \\ \quad \quad y_0 = 0 \\ \quad \quad z_0 = 0 \\ \hline \end{array}$$

$$\begin{array}{l} k=1 \quad x_1 = 1 \\ \quad \quad y_1 = -1 \\ \quad \quad z_1 = \frac{7}{4} \\ \hline \end{array}$$

$$\begin{array}{l} k=2 \quad x_2 = \frac{5}{8} \\ \quad \quad y_2 = -\frac{27}{16} \\ \quad \quad z_2 = \frac{129}{64} \\ \hline \end{array}$$

$$k=3$$

$$x = 1 - \frac{1}{2}y - \frac{1}{2}z$$

$$y = -\frac{1}{2} - \frac{1}{2}x - \frac{1}{2}z$$

$$z = \frac{7}{4} - \frac{1}{4}x - \frac{1}{4}y$$

$$\rightarrow 1 \quad ?$$

$$\rightarrow -2 \quad ?$$

$$\rightarrow 2 \quad \checkmark$$

Perhaps converges

$$\begin{aligned}x + 2y + 2z &= 1 \\x + y + z &= 1 \\x - y + z &= 5\end{aligned}$$



$$\begin{cases}x + y + z = 1 \rightarrow x = \\x + 2y + 2z = 1 \rightarrow y = \\x - y + z = 5 \rightarrow z =\end{cases}$$

• Gauss-Seidel

$$\begin{aligned}k=0 \quad x_0 &= 2 \\y_0 &= 0 \\z_0 &= 0\end{aligned}$$

$$\begin{aligned}k=1 \quad x_1 &= 1 \\y_1 &= 0 \\z_1 &= 4\end{aligned}$$

$$\begin{aligned}k=2 \quad x_2 &= -3 \\y_2 &= -2 \\z_2 &= 6\end{aligned}$$

elimination

$$\vec{x}_0 = (2, 0, 0)$$

2 iterations

$$\begin{aligned}x &= 1 - y - z \\y &= \frac{1}{2} - \frac{1}{2}x - z \\z &= 5 - x + y\end{aligned} \quad \left\{ \begin{aligned}x_{k+1} &= 1 - y_k - z_k \\y_{k+1} &= \frac{1}{2} - \frac{1}{2}x_{k+1} - z_k \\z_{k+1} &= 5 - x_{k+1} + y_{k+1}\end{aligned} \right.$$

relaxation ( $w$ )

$$\left[ \begin{array}{l} x_{k+1} = w \cdot (1 - y_k - z_k) + (1-w) x_k \\ y_{k+1} = w \cdot \left( \frac{1}{2} - \frac{1}{2} x_{k+1} - z_k \right) + (1-w) y_k \\ z_{k+1} = w \cdot (5 - x_{k+1} + y_{k+1}) + (1-w) z_k \end{array} \right]$$

$$x_{k+1} = \varphi(x_k)$$

$$x_{k+1} = \lambda \cdot \varphi(x_k) + (1-\lambda) x_k$$

$\downarrow$

$\lambda_{opt}$

$$\hookrightarrow \varphi_{\lambda}(x) = 0$$

$$\left[ \rho(w \cdot B_{GS} + (1-w) \cdot E_n) < 1 \quad \left( \begin{array}{l} \text{as} \\ \text{small} \\ \text{as possible} \end{array} \right) \right]$$

$w_{opt} \dots$  experiments

$\vec{x}_0 \rightarrow \rightarrow \rightarrow \vec{x}_k$  reliability / quality (?) (test of 3)

$$A \vec{x} = \vec{b} \rightsquigarrow \vec{r} = \vec{b} - A \vec{x}$$

$$\begin{cases} x + 4y - z = 12 \\ x + 2y + 4z = -2 \\ 2x - y + z = 0 \end{cases} \rightarrow \text{reorder the system} \\ \text{to increase the chance} \\ \text{that iterative methods will} \\ \text{work}$$

$$\begin{cases} 2x - y + z = 0 \\ x + 4y - z = 12 \\ x + 2y + 4z = -2 \end{cases} \rightarrow \text{derive iterative formulas}$$

$\rightarrow$  apply Jacobi method  
with  $\vec{x}_0 = (0, 0, 0)$ , show  
2 iterations

$\rightarrow$  apply the Gauss-Seidel method  
with  $\vec{x}_0 = (0, 0, 0)$ , show  
2 iterations

$$x = \frac{1}{2}y - \frac{1}{2}z$$

$$y = 3 - \frac{1}{4}x + \frac{1}{4}z$$

$$z = -\frac{1}{2} - \frac{1}{4}x - \frac{1}{2}y$$

• Jacobi

	$x = \frac{1}{2}y - \frac{1}{2}z$	$y = 3 - \frac{1}{4}x + \frac{1}{4}z$	$z = -\frac{1}{2} - \frac{1}{4}x - \frac{1}{2}y$
$k=0$	0	0	0
$k=1$	0	3	$-\frac{1}{2}$
$k=2$	$\frac{3}{2} + \frac{1}{4} = \frac{7}{4}$	$3 - 0 - \frac{1}{8} = \frac{23}{8}$	$-\frac{1}{2} - 0 - \frac{3}{2} = -2$

$$x_{k+1} = \frac{1}{2}y_k - \frac{1}{2}z_k$$

$$y_{k+1} = 3 - \frac{1}{4}x_k + \frac{1}{4}z_k$$

$$z_{k+1} = -\frac{1}{2} - \frac{1}{4}x_k - \frac{1}{2}y_k$$

# • Gauß-Seidel

$$k=0 \quad x_0 = 0$$

$$y_0 = 0$$

$$z_0 = 0$$

$$k=1 \quad x_1 = 0$$

$$y_1 = 3$$

$$z_1 = -\frac{1}{2} - \frac{3}{2} = -2$$

$$k=2 \quad x_2 = \frac{3}{2} + 1 = \frac{5}{2}$$

$$y_2 = 3 - \frac{5}{8} - \frac{1}{2} = \frac{15}{8}$$

$$z_2 = -\frac{1}{2} - \frac{5}{8} - \frac{15}{16} = -\frac{33}{16}$$

$$x = \frac{1}{2}y - \frac{1}{2}z$$

$$y = 3 - \frac{1}{4}x + \frac{1}{4}z$$

$$z = -\frac{1}{2} - \frac{1}{4}x - \frac{1}{2}y$$

sol.

$$x = 2$$

$$y = 2$$

$$z = -2$$

$$x_{k+1} = \frac{1}{2}y_k - \frac{1}{2}z_k$$

$$y_{k+1} = 3 - \frac{1}{4}x_{k+1} + \frac{1}{4}z_k$$

$$z_{k+1} = -\frac{1}{2} - \frac{1}{4}x_{k+1} - \frac{1}{2}y_{k+1}$$

$$x_{k+1} = \omega \left( \frac{1}{2} y_k - \frac{1}{2} z_k \right) + (1-\omega) x_k$$

$$y_{k+1} = \omega \left( 3 - \frac{1}{4} x_{k+1} + \frac{1}{4} z_k \right) + (1-\omega) y_k$$

$$z_{k+1} = \omega \left( -\frac{1}{2} - \frac{1}{4} x_{k+1} - \frac{1}{2} y_{k+1} \right) + (1-\omega) z_k$$

$$\omega_{\text{opt}} \approx 0.96$$