

## DMA Practice problems: Recurrence equations

**Exercise 1:** Find general solutions of the following equations:

- (i)  $a_{n+2} - 6a_{n+1} + 8a_n = 0, n \geq 0;$
- (ii)  $a_{n+2} - 4a_n = 0, n \geq 1;$
- (iii)  $a_{n+2} + a_{n+1} - 2a_n = 0, n \geq 2;$
- (iv)  $a_{n+2} - 6a_{n+1} + 9a_n = 0, n \geq -2;$
- (v)  $a_{n+2} + 9a_n = 0, n \geq 0;$
- (vi)  $a_{n+3} - 2a_{n+2} - a_{n+1} + 2a_n = 0, n \geq 1;$
- (vii)  $a_{n+2} - a_n = 18n 2^n, n \geq 0;$
- (viii)  $a_{n+2} + 2a_{n+1} - 3a_n = (5n + 12)2^n, n \geq 0;$
- (ix)  $a_{n+2} - 4a_{n+1} + 4a_n = 13 \cdot 3^n - 3, n \geq 1;$
- (x)  $a_{n+1} = a_n + 2a_{n-1} + 3 \cdot 2^n - 2 \cdot (-2)^n, n \geq 2;$
- (xi)  $a_{n+3} + 3a_{n+2} - 4a_n = 16 \cdot 2^n + 9, n \geq 0;$
- (xii)  $a_{n+1} = a_{n-1} + n - 1, n \geq 2.$

**Exercise 2:** Find solutions of the following initial value problems:

- (i)  $5a_n = a_{n+1} + 6a_{n-1}, n \geq 2; a_1 = 9, a_2 = 21;$
- (ii)  $a_{n+2} - 4a_n = 0, n \geq 1; a_1 = 2, a_2 = 0.$
- (iii)  $a_{n+1} = 6a_{n-1} - a_n, n \geq 3; a_2 = -6, a_3 = 78;$
- (iv)  $a_{n+1} = 4a_n - 5a_{n-1} + 2a_{n-2}, n \geq 2; a_0 = 1, a_1 = 1, a_2 = 2;$
- (v)  $a_{n+2} - a_n = (8n + 18)3^n, n \geq 1; a_1 = 16, a_2 = 31;$
- (vi)  $a_{n+1} = 2a_n - a_{n-1} - 4 \cdot (-1)^n, n \geq 0; a_{-1} = 0, a_0 = 2;$
- (vii)  $a_{n+2} + 2a_{n+1} - 3a_n = 5 \cdot 2^n + 8, n \geq 1; a_1 = 5, a_2 = 9;$
- (viii)  $a_{n+1} = a_n + 4a_{n-1} - 4a_{n-2} + (6n - 7)(-1)^n, n \geq 2; a_0 = 2, a_1 = 4, a_2 = 7.$
- (ix)  $a_{n+2} + 4a_n = 0, n \geq 0; a_0 = 0, a_1 = 1;$

**Solution 1:** (i):  $(\lambda - 2)(\lambda - 4) = 0$ ,  $\{2^n u + 4^n v\}_{n=0}^\infty$ ; (ii):  $(\lambda - 2)(\lambda + 2) = 0$ ,  $\{2^n u + (-2)^n v\}_{n=1}^\infty$ ;  
 (iii):  $(\lambda + 2)(\lambda - 1) = 0$ ,  $\{u + (-2)^n v\}_{n=2}^\infty$ ; (iv):  $(\lambda - 3)^2 = 0$ ,  $\{(un + v)3^n\}_{n=-2}^\infty$ ;  
 (v):  $(\lambda - 3i)(\lambda + 3i) = 0$ ,  $\{(u \cos(n\frac{\pi}{2}) + v \sin(n\frac{\pi}{2}))3^n\}_{n=0}^\infty$ ;  
 (vi):  $(\lambda - 1)(\lambda + 1)(\lambda - 2) = 0$ ,  $\{u + (-1)^n v + 2^n w\}_{n=1}^\infty$ ;  
 (vii):  $(\lambda - 1)(\lambda + 1) = 0$ ,  $a_{h,n} = u + (-1)^n v$ ; guess  $a_n = (An + B)2^n$ ,  $\{(6n - 16)2^n + u + (-1)^n v\}_{n=0}^\infty$ ;  
 (viii):  $(\lambda - 1)(\lambda + 3) = 0$ ,  $a_{h,n} = u + (-3)^n v$ ; guess  $a_n = (An + B)2^n$ ,  $\{n2^n + u + (-3)^n v\}_{n=0}^\infty$ ;  
 (ix):  $(\lambda - 2)^2 = 0$ ,  $a_{h,n} = n2^n u + 2^n v$ ; guess  $a_n = A3^n + B$ ,  $\{13 \cdot 3^n - 3 + n2^n u + 2^n v\}_{n=1}^\infty$ ;  
 (x): rewrite:  $a_{n+2} - a_{n+1} - 2a_n = 6 \cdot 2^n + 4 \cdot (-2)^n$ ,  $n \geq 1$ ;  $(\lambda - 2)(\lambda + 1) = 0$ ,  $a_{h,n} = 2^n u + (-1)^n v$ ; guess  $a_n = n^1 A2^n + B(-2)^n$ ,  $\{n2^n + (-2)^n + 2^n u + (-1)^n v\}_{n=1}^\infty$ ;  
 (xi):  $(\lambda - 1)(\lambda + 2)^2 = 0$ ,  $a_{h,n} = u + (-2)^n v + n(-2)^n w$ ; guess  $a_n = A2^n + n^1 B$ ,  $\{n + 2^n + u + (-2)^n v + n(-2)^n w\}_{n=0}^\infty$ ;  
 (xii): rewrite:  $a_{n+2} - a_n = n = n \cdot 1^n$ ,  $n \geq 1$ ;  $(\lambda - 1)(\lambda + 1) = 0$ ,  $a_{h,n} = u + (-1)^n v$ ; guess  $a_n = n(An + B)$ ,  $\{\frac{1}{4}n^2 - \frac{1}{2}n + u + (-1)^n v\}_{n=0}^\infty$ .

**Solution 2:** (i): We rewrite:  $a_{n+2} - 5a_{n+1} + 6a_n = 0$ ,  $n \geq 1$ ;  $(\lambda - 2)(\lambda - 3) = 0$ , general sol.  $\{2^n u + 3^n v\}_{n=1}^\infty$ , init. conditions yield  $\{3 \cdot 2^n + 3^n\}_{n=1}^\infty$ ;  
 (ii):  $(\lambda - 2)(\lambda + 2) = 0$ , general sol.  $\{2^n u + (-2)^n v\}_{n=1}^\infty$ ; init. conditions yield  $\{2^{n-1} + (-2)^{n-1}\}_{n=1}^\infty$ , it goes  $\{2, 0, 8, 0, 32, 0, 64, \dots\}$ .  
 (iii): rewrite:  $a_{n+2} + a_{n+1} - 6a_n = 0$ ,  $n \geq 2$ ;  $(\lambda - 2)(\lambda + 3) = 0$ ,  $\{2^n u + (-3)^n v\}_{n=2}^\infty$ , init. cond. give  $\{3 \cdot 2^n - 2(-3)^n\}_{n=2}^\infty$ ;  
 (iv): rewrite:  $a_{n+3} - 4a_{n+2} + 5a_{n+1} - 2a_n = 0$ ,  $n \geq 0$ ;  $(\lambda - 1)^2(\lambda - 2) = 0$ ;  $\{u + nv + 2^n w\}_{n=0}^\infty$ ; init. cond. give  $\{2^n - n\}_{n=0}^\infty$ ;  
 (v):  $(\lambda - 1)(\lambda + 1) = 0$ ,  $a_{h,n} = u + (-1)^n v$ ; guess  $a_n = (An + B)3^n$ ,  $\{n3^n + u + (-1)^n v\}_{n=1}^\infty$ , init. cond. give  $\{n3^n + 13\}_{n=1}^\infty$ ;  
 (vi): rewrite:  $a_{n+2} - 2a_{n+1} + a_n = 4 \cdot (-1)^n$ ,  $n \geq -1$ ;  $(\lambda - 1)^2 = 0$ ,  $a_{h,n} = u + nv$ ; guess  $a_n = A(-1)^n$ ,  $\{(-1)^n + u + nv\}_{n=-1}^\infty$ , init. cond. give  $\{(-1)^n + 1\}_{n=-1}^\infty$ , it is  $\{0, 2, 0, 2, 0, 2, 0, 2, \dots\}$ ;  
 (vii):  $(\lambda - 1)(\lambda + 3) = 0$ ,  $a_{h,n} = u + (-3)^n v$ ; guess  $a_n = A2^n + n^1 B$ ,  $\{2^n + 2n + u + (-3)^n v\}_{n=1}^\infty$ , init. cond. give  $\{2^n + 2n + 1\}_{n=1}^\infty$ ;  
 (viii): rewrite:  $a_{n+3} - a_{n+2} - 4a_{n+1} + 4a_n = (6n + 5)(-1)^n$ ,  $n \geq 0$ ;  $(\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$ ,  $a_{h,n} = u + 2^n v + (-2)^n w$ ; guess  $a_n = (An + B)(-1)^n$ ,  $\{(n + 1)(-1)^n + u + 2^n v + (-2)^n w\}_{n=0}^\infty$ , init. cond. give  $\{(n + 1)(-1)^n + 2^{n+1} - (-2)^n\}_{n=0}^\infty$ .  
 (ix):  $(\lambda - 2i)(\lambda + 2i) = 0$ ,  $\{(u \cos(n\frac{\pi}{2}) + v \sin(n\frac{\pi}{2}))2^n\}_{n=0}^\infty$ ; init. cond. give  $\{2^{n-1} \sin(n\frac{\pi}{2})\}_{n=0}^\infty$ , it is  $\{0, 1, 0, -4, 0, 16, 0, -64, 0, \dots\}$ ;