DMA Practice problems: Relations

Exercise 1: For the following relations on $\mathbb{Z}$, investigate the four basic properties.

(i) $a\mathcal{R}b$ iff $|a| = |b|$;
(ii) $a\mathcal{R}b$ iff $a \geq b$;
(iii) $a\mathcal{R}b$ iff $a \neq b$;
(iv) $a\mathcal{R}b$ iff $a = b + 1$;
(v) $a\mathcal{R}b$ iff $a - b = 2k$ for some $k \in \mathbb{Z}$;
(vi) $a\mathcal{R}b$ iff a and b share some common divisor other than 1;
(vii)* $a\mathcal{R}b$ iff $a \geq b^2$ (see the next exercise);
(viii)* $a\mathcal{R}b$ iff $2a \leq b$.

Exercise 2: For the following relations on $\mathbb{R}$, investigate the four basic properties.

(i) $x\mathcal{R}y$ iff $y - x \in \mathbb{Z}$;
(ii) $x\mathcal{R}y$ iff $x - y \in \mathbb{Q}$;
(iii) $x\mathcal{R}y$ iff $xy \geq 0$;
(iv) $x\mathcal{R}y$ iff $xy \geq 1$;
(v) $x\mathcal{R}y$ iff $x = y^2$;
(vi)* $x\mathcal{R}y$ iff $x \geq y^2$ (see the previous exercise);
(vii) $x\mathcal{R}y$ iff $|x| \leq |y|$.

Exercise 3: Investigate the basic four properties for the following relations:

(i) Relation $\mathcal{R}$ on the set $\mathbb{R}^2$ defined as follows: $(u, v)\mathcal{R}(x, y)$ iff $u^2 - y = x^2 - v$, formally, $\mathcal{R} = \{((u, v), (x, y)) \in \mathbb{R}^2 \times \mathbb{R}^2;\ u^2 - y = x^2 - v\}$.

(ii) Relation $\mathcal{R}$ on the set $\mathbb{R}^2$ defined as follows: $(u, v)\mathcal{R}(x, y)$ iff $u^2 - y = v^2 - x$, formally, $\mathcal{R} = \{((u, v), (x, y)) \in \mathbb{R}^2 \times \mathbb{R}^2;\ u^2 - y = x^2 - v\}$.

(iii)* Relation $\mathcal{R}$ on the set $\mathbb{R}^2$ defined as follows:
Consider the set $N = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 = 13\}$ (incidentally, it is the circle around the origin with radius $\sqrt{13}$). Define $\mathcal{R} = \{((u, v), (x, y)) \in \mathbb{R}^2 \times \mathbb{R}^2;\ (u, v) - (x, y) \in N\}$.

(iv)* Relation $\mathcal{R}$ on the set $\mathbb{R}^2$ defined as follows:
Consider the set $N = \{(x, y) \in \mathbb{R}^2;\ x + y = 0\}$ (incidentally, it is the antidiagonal or secondary diagonal). Define $\mathcal{R} = \{((u, v), (x, y)) \in \mathbb{R}^2 \times \mathbb{R}^2;\ (u, v) - (x, y) \in N\}$.

(v) Relation $\mathcal{R}$ on the set F of all mappings $\mathbb{Z} \mapsto \mathbb{Z}$ defined as $T\mathcal{R}S$ iff $T(0)S(0) = 2$.

(vi) Relation $\mathcal{R}$ on the set F of all mappings $\mathbb{Z} \mapsto \mathbb{Z}$ defined as $T\mathcal{R}S$ iff $T(1) = S(2)$.

(vii) Relation $\mathcal{R}$ on the set F of all functions $\mathbb{R} \mapsto \mathbb{R}$ defined as $f\mathcal{R}g$ iff $f(x) \geq g(y)$ for all $x \in \mathbb{R}$.

(viii) Relation $\mathcal{R}$ on the set $M_{2\times 2}$ of all 2×2 real matrices defined as $A\mathcal{R}B$ iff $|A| = |B|$ (the same determinant).

(ix) Relation $\mathcal{R}$ on the set $M_{2\times 2}$ all 2×2 real matrices defined as $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathcal{R} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ iff $a_{11} = b_{22}$.

(x) Relation $\mathcal{R}$ on the set P of all real polynomials defined as $p\mathcal{R}q$ iff p and q have the same degree.

(xi) Relation $\mathcal{R}$ on the set P all real polynomials defined as $p\mathcal{R}q$ iff p and q have the same roots including their multiplicities.

(xii) Relation $\mathcal{R}$ on the set P all real polynomials defined as $p\mathcal{R}q$ iff p and q have the same complex roots including their multiplicities.

Solution 1: (i): **R**: For every $a \in \mathbb{Z}$ we have $|a| = |a|$, hence $a\mathcal{R}a$. Is reflexive.
S: Arbitrary $a, b \in \mathbb{Z}$ satisfying $a\mathcal{R}b$, that gives $|a| = |b|$, hence $|b| = |a|$ and so $b\mathcal{R}a$. Is symmetric.
A: Arbitrary $a, b \in \mathbb{Z}$ satisfying $a\mathcal{R}b$ and $b\mathcal{R}a$, that gives $|a| = |b|$ and $|b| = |a|$, we will not get $a = b$ from this. Counterexample: $|-13| = |13|$, hence $13\mathcal{R}(-13)$ and $(-13)\mathcal{R}13$, but $-13 = 13$ not true, so $\mathcal{R}$ is not antisymmetric.
T: Arbitrary $a, b, c \in \mathbb{Z}$ satisfying $a\mathcal{R}b$ and $b\mathcal{R}c$, that gives $|a| = |b|$ and $|b| = |c|$, from that we have $|a| = |c|$ and so $a\mathcal{R}c$. $\mathcal{R}$ is transitive.
(ii): **R**: yes, for $a \in A$ we have $a \geq a$, hence $a\mathcal{R}a$; **S**: no, $2\mathcal{R}1$ as $2 \geq 1$, but $1 \geq 2$ not true, hence $1\mathcal{R}2$ not true;
A: yes, $a\mathcal{R}b \wedge b\mathcal{R}a \implies a \geq b \wedge b \geq a \implies a = b$; **T**: yes, $a\mathcal{R}b \wedge b\mathcal{R}c \implies a \geq b \wedge b \geq c \implies a \geq c \implies a\mathcal{R}c$.
(iii): **R**: no, for instance $1 \neq 1$ not true hence $1R1$ not true; **S**: yes, $a\mathcal{R}b \implies a \neq b \implies b \neq a \implies b\mathcal{R}a$;
A: no, say, $1\mathcal{R}2 \wedge 2\mathcal{R}1$, but $1 = 2$ not true; **T**: no, say, $1\mathcal{R}2$ and $2\mathcal{R}1$, but $1\mathcal{R}1$ not true.
(iv): **R**: no, $13 = 13 + 1$ not true and hence $13\mathcal{R}13$ not true; **S**: no, $2\mathcal{R}1$ but $1\mathcal{R}2$ not true;
A: yes, $a\mathcal{R}b \wedge b\mathcal{R}a \implies a = b + 1 \wedge b = a + 1 \implies b = b + 2 \implies 0 = 2$ contradiction, so the assumption is never true, hence the implication is always valid; **T**: no, say, $3\mathcal{R}2$ and $2\mathcal{R}1$, but $3\mathcal{R}1$ not true.
(v): **R**: yes, $a - a = 2 \cdot 0 \implies a\mathcal{R}a$ for every a; **S**: yes, $a\mathcal{R}b \implies a - b = 2k \implies b - a = 2(-k) \implies b\mathcal{R}a$;
A: no, say, $1\mathcal{R}3$ and $3\mathcal{R}1$, yet $1 = 3$ not true;
T: yes, $a\mathcal{R}b \wedge b\mathcal{R}c \implies a - b = 2k \wedge b - c = 2l \implies a - c = 2(k + l) \implies a\mathcal{R}c$.
(vi): **R**: Does every $a \in \mathbb{Z}$ have some common divisor with itself other than 1? Almost yes, not true for $a = 1$. So $\mathcal{R}$ is not reflexive.
S: Let $a, b \in \mathbb{Z}$ satisfy $a\mathcal{R}b$. Then there is $c > 1$ that divides both a and b, it then also divides b and a, so $b\mathcal{R}a$. $\mathcal{R}$ is symmetric.
A: $a\mathcal{R}b \wedge b\mathcal{R}a$ gives a common divisor, no chance to force $a = b$. Counterexample: $2\mathcal{R}4$ and $4\mathcal{R}2$ (common divisor 2), hence is not antisymmetric.
T: a, b have common divisor > 1, b, c have common divisor > 1, this does not yield anything common for a, c. Counterexample: $2\mathcal{R}6$ and $6\mathcal{R}3$, but not $2\mathcal{R}3$. It is not transitive.
(vii): Is not **R**, see $a = 2$; not **S** see $4\mathcal{R}2$;
A: $a\mathcal{R}b \wedge b\mathcal{R}a \implies a \geq b^2 \wedge b \geq a^2$. If $a = 0$, then that gives $0 \geq b^2 \implies b = 0 = a$. If $a \neq 0$, then $|a| \geq 1$, also $a \geq b^2 \geq 0$ and hence $a \geq 1$, similarly $b \geq 1$. We calculate: $a \geq b^2 \wedge b \geq a^2 \implies a \geq b^2 \geq a^4 \implies a \geq a^4 \implies 1 \geq a^3$, together with $a \geq 1$ that gives $a = 1$. Then $1 \geq b^2 \geq 1 \implies b = 1$ and again $a = b$. Relation is antisymmetric.
T: For $b \in \mathbb{Z}$ we have $b^2 \geq b$ (see A), hence $a\mathcal{R}b \wedge b\mathcal{R}c \implies a \geq b^2 \wedge b \geq c^2 \implies a \geq b \geq c^2 \implies a \geq c^2 \implies a\mathcal{R}c$. is transitive.
(viii): **R**: no, inequality $2a \leq a$ is valid only for negative a and zero, counterexample $a = 1$;
S: no, $a\mathcal{R}b \implies 2a \leq b$, this gives $2b \geq 4a$, but we need $2b \leq a$. Counterexample $a = 1$, $b = 2$.
A: no, $[a\mathcal{R}b \wedge b\mathcal{R}a] \implies [2a \leq b \wedge 2b \leq a] \implies [4a \leq 2b \wedge 2b \leq a]$, so $4a \leq a$. This could happen for non-positive a, we will look for counterexample there. We find, say, $a = -3$ and $b = -2$. Remark: A would be true on $\mathbb{N}$.
T: no, $[a\mathcal{R}b \wedge b\mathcal{R}c] \implies [2a \leq b \wedge 2b \leq c] \implies 4a \leq c$. For $a \geq 0$ we have $2a \leq 4a \leq c$, so $2a \leq c$ and $a\mathcal{R}c$. On $\mathbb{N}$ we would have transitivity. But we also have negative numbers, counterexample $a = -1$, $b = -2$, $c = -4$.

Solution 2: (i): **R**,**S**,**T**, see example in the book;
(ii): **R** yes $x - x = 0 \in \mathbb{Q}$, **S** yes $y - x \in \mathbb{Q} \implies x - y = -(y - x) \in \mathbb{Q}$, **T** yes $y - x \in \mathbb{Q} \wedge (z - y) \in \mathbb{Q}$
$\implies (z - x) = (y - x) + (z - y) \in \mathbb{Q}$; not **A** see $1\mathcal{R}2$ and $2\mathcal{R}1$;

(iii): **R** yes $xx=x^2\geq 0$, **S** yes $xy\geq 0 \implies yx\geq 0$; not **A** see $1\mathcal{R}2$ and $2\mathcal{R}1$; not **T** see $(-1)\mathcal{R}0$ and $0\mathcal{R}1$;
(iv): Not **R** see $x=0$, **S** yes $xy\geq 1 \implies yx\geq 1$; not **A** see $2\mathcal{R}1$ and $1\mathcal{R}2$; not **T** see $\frac{1}{2}\mathcal{R}4$ and $4\mathcal{R}1$;
(v): Not **R** see $x=2$; not **S** see $4\mathcal{R}2$;
A yes $x=y^2\wedge y=x^2 \implies x,y\geq 0\wedge x=x^4\wedge y=y^4 \implies x=y=1\vee x=y=0$; not **T** see $16\mathcal{R}4$ and $4\mathcal{R}2$;
(vi): Not **R** see $x=2$; not **S** see $4\mathcal{R}2$; not **A** see $x=0.1$, $y=0.2$ as $0.1\geq(0.2)^2$ and $0.2\geq(0.1)^2$ but not $0.1=2$; not **T** see $(0.5)\mathcal{R}(0.7)$ as $0.5\geq(0.7)^2=0.49$, $(0.7)\mathcal{R}(0.8)$ as $0.7\geq 0.64$, but not $0.5\geq 0.64$ (this was probably a bit tricky).
(vii): **R** yes $|x|\leq|x|$, **T** yes $|x|\leq|y|\wedge|y|\leq|z| \implies |x|\leq|z|$; not **S** see $1\mathcal{R}2$, not **A** see $1\mathcal{R}(-1)$ and $(-1)\mathcal{R}2$.

Solution 3: (i): **R**: yes $u^2-v=u^2-v \implies (u,v)R(u,v)$; **S**: $(u,v)\mathcal{R}(x,y)$ $\implies u^2-y=x^2-v \implies x^2-v=u^2-y \implies (x,y)\mathcal{R}(u,v)$ yes;
A: no, see e.g. $(1,4)\mathcal{R}(2,1)$ and $(2,1)\mathcal{R}(1,4)$;
T: yes; $(s,t)\mathcal{R}(u,v)$ & $(u,v)\mathcal{R}(x,y) \implies s^2-v=u^2-t\wedge u^2-y=x^2-v$ add equations, $s^2-v+u^2-y=u^2-t+x^2-v \implies s^2-y=x^2-t \implies (s,t)\mathcal{R}(x,y)$.
(ii): **R**: no, see e.g. $(2,3)$, not true that $2^2-3=3^2-2$; **S**: no, see e.g. $(2,1)\mathcal{R}(1,4)$ but not $(1,4)\mathcal{R}(2,1)$; **A**: no, see e.g. $(1,0)\mathcal{R}(0,1)$ and $(0,1)\mathcal{R}(1,0)$; **T**: no, see e.g. $(1,4)\mathcal{R}(2,1)$ and $(2,1)\mathcal{R}(1,4)$ but not $(1,4)\mathcal{R}(1,4)$.
(iii): rewrite: $(u,v)\mathcal{R}(x,y) \iff (u-x)^2+(v-y)^2=13$; **R**: no $(u-u)^2+(v-v)^2=0\neq 13$; **S**: yes $(u,v)\mathcal{R}(x,y) \implies (u-x)^2+(v-y)^2=13 \implies (x-u)^2+(y-v)^2=13 \implies (x,y)\mathcal{R}(u,v)$; **A**: no, say, $(4,3)\mathcal{R}(1,1)$ and $(1,1)\mathcal{R}(4,3)$; **T**: no, say, $(4,3)\mathcal{R}(1,1)$ and $(1,1)\mathcal{R}(4,3)$ but not $(4,3)\mathcal{R}(4,3)$.
(iv): rewrite: $(u,v)\mathcal{R}(x,y) \iff (u-x)+(v-y)=0$; **R**: yes $(u-u)+(v-v)=0$;
S: yes $(u,v)\mathcal{R}(x,y) \implies (u-x)+(v-y)=0 \implies (x-u)+(y-v)=0 \implies (x,y)\mathcal{R}(u,v)$;
A: no, say, $(1,3)\mathcal{R}(2,2)$ and $(2,2)\mathcal{R}(1,3)$; **T**: yes $(s,t)\mathcal{R}(u,v)\wedge(u,v)\mathcal{R}(x,y) \implies (s-u)+(t-v)=0\wedge(u-x)+(v-y)=0$ add equations, $(s-x)+(t-y)=0 \implies (s,t)\mathcal{R}(x,y)$.
(v): **R**: no, this would require that all mappings satisfy $T(0)T(0)=2$, but for instance the mapping $T(n)=n+1$ has $T(0)T(0)=1\cdot 1=1$;
S: yes $T\mathcal{R}S \implies T(0)S(0)=2 \implies S(0)T(0)=2 \implies S\mathcal{R}T$; **A**: no, say, $T(n)=n+1$, $S(n)=3n+2$, then $T(0)S(0)=1\cdot 2=2=S(0)T(0)$, so $T\mathcal{R}S$ and $S\mathcal{R}T$, but not $T=S$; **T**: no, say, $T(n)=n+1$, $S(n)=3n+2$, $U(n)=(n+1)^2$, then $T\mathcal{R}S$ and $S\mathcal{R}U$, but not $T\mathcal{R}U$ as $T(0)U(0)=1$.
(vi): **R**: no, this would require that all mappings satisfy $T(1)=T(2)$, but for instance the mapping $T(n)=n$ has $T(1)=1$ and $T(2)=2$;
S: no, say, $T(n)=n+1$ and $S(n)=n$, then $T(1)=1=S(2)$, hence $T\mathcal{R}S$, but $S(1)=T(2)$ not true; **A**: no, say, $T(n)=(2n-3)^2$, $S(n)=1$ (a constant mapping), then $T(1)=1=S(2)$ and $S(1)=1=T(2)$, hence $T\mathcal{R}S$ and $S\mathcal{R}T$, but not $T=S$; **T**: no, say, $T(n)=n+1$, $S(n)=n$, $U(n)=n-1$, then $T\mathcal{R}S$ and $S\mathcal{R}U$, but not $T\mathcal{R}U$ as $T(1)=2$ and $U(2)=1$.
(vii): **R**: yes, arbitrary function f satisfies the inequality $f(x)\geq f(x)$ for all $x\in\mathbb{R}$; **S**: no, say, $f(x)=x+13$, $g(x)=x$ satisfy $f\mathcal{R}g$ but not $g\mathcal{R}f$; **A**: yes, $f\mathcal{R}g$ and $g\mathcal{R}f$ mean $f(x)\geq g(x)$ and $g(x)\geq f(x)$ for all x, that is, $f(x)=g(x)$ for all x, that is, $f=g$; **T**: yes, $f\mathcal{R}g$ and $g\mathcal{R}h$ give for all $x\in\mathbb{R}$ that $f(x)\geq g(x)$ and $g(x)\geq h(x)$, that is, $f(x)\geq h(x)$, so $f\mathcal{R}h$.
(viii): **R**: yes $|A|=|A|$; **S**: yes $A\mathcal{R}B \implies |A|=|B| \implies |B|=|A| \implies B\mathcal{R}A$; **A**: no, say, a matrix of all zeros or a non-zero matrix with repeated rows have zero determinant; **T**: yes $A\mathcal{R}B\wedge B\mathcal{R}C \implies |A|=|B|\wedge|B|=|C| \implies |A|=|C| \implies A\mathcal{R}C$.
(ix): **R**: no, say, in the matrix $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ the upper left and lower bottom corners do not match, hence $A\mathcal{R}A$ not true;

S: no, say, for $A = \begin{pmatrix} 13 & 2 \\ -2 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 3 & 13 \end{pmatrix}$ we have $A\mathcal{R}B$, but not $B\mathcal{R}A$;
A: no, say, $A = \begin{pmatrix} 13 & 1 \\ -1 & 23 \end{pmatrix}$ and $B = \begin{pmatrix} 23 & 2 \\ 3 & 13 \end{pmatrix}$ satisfy $A\mathcal{R}B$ and $B\mathcal{R}A$, but not $A = B$;
T: no, say, $A = \begin{pmatrix} 13 & 1 \\ -1 & 23 \end{pmatrix}$, $B = \begin{pmatrix} 23 & 2 \\ 3 & 13 \end{pmatrix}$ and $C = \begin{pmatrix} 14 & -3 \\ 5 & 23 \end{pmatrix}$ satisfy $A\mathcal{R}B$ and $B\mathcal{R}C$, but not $A\mathcal{R}C$.
(x): **R**: yes $\deg(p) = \deg(p)$; **S**: yes $p\mathcal{R}q \implies \deg(p) = \deg(q) \implies \deg(q) = \deg(p) \implies q\mathcal{R}p$; **A**: no, say, $p = x$ and $q = 2x+1$; **T**: yes $p\mathcal{R}q \wedge q\mathcal{R}r \implies \deg(p) = \deg(q) \wedge \deg(q) = \deg(r) \implies \deg(p) = \deg(r) \implies p\mathcal{R}r$;
(xi): **R**: yes; **S**: yes; **A**: no, say, $p = x - 1$ and $q = 2x - 2$; **T**: yes;
(xii): **R**: yes; **S**: yes; **A**: no, say, $p = x - 1$ and $q = 2x - 2$; **T**: yes;