DMA Practice problems: Induction

Exercise 1: Prove by induction that the following formulas are true for all $n \in \mathbb{N}$:
(i) $2 + 4 + 6 + \cdots + (2n) = n(n+1)$;
(ii) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$;
(iii) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$;
(iv) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)\cdot(2n+1)} = \frac{n}{2n+1}$;
(v) $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$;
(vi) $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 2 - \frac{1}{n!}$;
(vii) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$;
(viii) $n! < n^n$ (this one for $n \geq 2$).

Exercise 2: Consider functions defined inductively by the following formulas. For each of them calculate the first few values and try to guess an explicit formula for $f(n)$. Then prove that the formula is correct.
(i) (0) $f(0) = 0$, (1) $f(n+1) = 2f(n)$ for $n \in \mathbb{N}_0$;
(ii) (0) $f(1) = 0$, (1) $f(n+1) = f(n) + 1$ for $n \in \mathbb{N}$;
(iii) (0) $f(1) = 1$, (1) $f(n+1) = f(n) \cdot \frac{n}{n+1}$ for $n \in \mathbb{N}$;
(iv) (0) $f(1) = 1$, $f(2) = 2$, (1) $f(n+1) = 2f(n) - f(n-1)$ for $n \in \mathbb{N}$, $n \geq 2$;
(v) (0) $f(1) = 1$, $f(2) = 1$, $f(3) = 1$, (1) $f(n+1) = f(n) + f(n-1) - f(n-2)$ for $n \in \mathbb{N}$, $n \geq 3$;
(vi) (0) $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, (1) $f(n+1) = f(n) + f(n-1) - f(n-2)$ for $n \in \mathbb{N}$, $n \geq 3$;
(vii) (0) $f(0) = 1$, $f(1) = 3$, (1) $f(n+1) = \begin{cases} 3f(n), & n \in \mathbb{N} \text{ odd;} \\ 9f(n-1), & n \in \mathbb{N} \text{ even;} \end{cases}$
(viii) (0) $f(1) = 1$, $f(2) = 2$, (1) $f(n+1) = 2f(n-1)$ for $n \in \mathbb{N}$, $n \geq 2$;

Exercise 3: Consider functions defined inductively by the following formulas. For each of them prove the given (in)equality.
(i) (0) $f(1) = 1$, $f(2) = 2$, (1) $f(n+1) = f(n) + n\,f(n-1)$ for $n \geq 2$; inequality $f(n) \leq n!$;
(ii) (0) $f(1) = 1$, $f(2) = 2$, (1) $f(n+1) = \frac{1}{n}f(n) + f(n-1)$ for $n \geq 2$; inequality $f(n) \leq n^2$;
(iii) (0) $f(1) = 1$, $f(2) = 2$, (1) $f(n+1) = n\,f(n) + n\,f(n-1)$ for $n \geq 2$; equality $f(n) = n!$;
(iv) (0) $f(1) = 2$, $f(2) = 3$, (1) $f(n+1) = n\,f(n) + n^2 f(n-1)$ for $n \geq 2$; inequality $f(n) \geq n!$.

Exercise 4: Define the set of all binary words (chains) that
(i) do not contain adjacent zeros;
(ii) end with a zero;
(iii) do not end with a zero;
(iv) contain the combination 101.

Exercise 5: Define the set of all words over the alphabet $C = \{1, 2, 3, 4\}$ that:
(i) do not contain adjacent threes;
(ii) start with a two;
(iii) do not end with a one;
(iv) have the same number of odd and even numerals.

Solution 1: (i): (0) $V(1)$ says $2=1{\cdot}2$, true. (1) Let $n\in\mathbb{N}$. Assumption: $2+4+6+\cdots+(2n)=n(n+1)$. To prove: $2+4+6+\cdots+(2n+2)=(n+1)(n+2)$. Decomposition:
$2+4+6+\cdots+(2n+2)=[2+4+6+\cdots+(2n)]+(2n+2)\overset{\text{IP}}{=\!=}[n(n+1)]+(2n+2)=n^2+3n+2=(n+1)(n+2)$.
(ii): (0) $V(1)$ says $1=\frac{1}{2}1\cdot 2$, true. (1) Let $n\in\mathbb{N}$. Assumption: $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$.
To prove: $1+2+3+\cdots+(n+1)=\frac{1}{2}(n+1)(n+2)$. Decomposition:
$1+2+3+\cdots+(n+1)=[1+2+3+\cdots+n]+(n+1)\overset{\text{IP}}{=\!=}\left[\frac{1}{2}n(n+1)\right]+(n+1)=\frac{1}{2}(n^2+3n+2)=\frac{1}{2}(n+1)(n+2)$.
(iii): (0) $V(1)$ says $1^2=\frac{1}{6}1\cdot 2\cdot 3$, true. (1) Let $n\in\mathbb{N}$. Assumption: $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.
To prove: $1^2+2^2+3^2+\cdots+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3)$. Decomposition:
$1^2+2^2+3^2+\cdots+(n+1)^2=[1^2+2^2+3^2+\cdots+n^2]+(n+1)^2\overset{\text{IP}}{=\!=}\left[\frac{1}{6}n(n+1)(2n+1)\right]+(n+1)^2=$
$=\frac{1}{6}(2n^3+9n^2+13n+6)=\frac{1}{6}(n+1)(n+2)(2n+3)$.
(iv): (0) $V(1)$ says $\frac{1}{3}=\frac{1}{3}$, true. (1) Let $n\in\mathbb{N}$. Assumption: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}$.
To prove: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n+1)\cdot(2n+3)}=\frac{n+1}{2n+3}$. Decomposition: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n+1)\cdot(2n+3)}=$
$=\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}\right]+\frac{1}{(2n+1)\cdot(2n+3)}\overset{\text{IP}}{=\!=}\left[\frac{n}{2n+1}\right]+\frac{1}{(2n+1)\cdot(2n+3)}=\frac{2n^2+3n+1}{(2n+1)(2n+3)}=\frac{n+1}{2n+3}$.
(v): (0) $V(1)$ says $1\cdot 1=2-1$, true. (1) Let $n\in\mathbb{N}$. Assumption: $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$.
To prove: $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!+(n+1)\cdot(n+1)!=(n+2)!-1$. Decomposition:
$1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!+(n+1)\cdot(n+1)!=[1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!]+(n+1)\cdot(n+1)!=$
$\overset{\text{IP}}{=\!=}[(n+1)!-1]+(n+1)\cdot(n+1)!=(n+1)!+(n+1)\cdot(n+1)!-1=(n+2)(n+1)!-1=(n+2)!-1$.
(vi): (0) $V(1)$ says $1\leq 2-1$, true. (1) Let $n\in\mathbb{N}$. Assumption: $1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\leq 2-\frac{1}{n!}$.
To prove: $1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n+1)!}\leq 2-\frac{1}{(n+1)!}$. Decomposition: $1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n+1)!}=$
$=\left[1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right]+\frac{1}{(n+1)!}\overset{\text{IP}}{\leq}\left[2-\frac{1}{n!}\right]+\frac{1}{(n+1)!}=2-\frac{(n+1)-1}{(n+1)!}=2-\frac{n}{(n+1)!}\leq 2-\frac{1}{(n+1)!}$.
(vii): (0) $V(1)$ says $1\leq 2-1$, true. (1) Let $n\in\mathbb{N}$. Assumption: $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}\leq 2-\frac{1}{n}$.
To prove: $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2}\leq 2-\frac{1}{n+1}$. Decomposition: $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2}$
$=\left[1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}\right]+\frac{1}{(n+1)^2}\overset{\text{IP}}{\leq}\left[2-\frac{1}{n}\right]+\frac{1}{(n+1)^2}=2-\frac{(n+1)^2-n}{n(n+1)^2}=2-\frac{n^2+n+1}{n(n+1)^2}\leq 2-\frac{n^2+n}{n(n+1)^2}=2-\frac{1}{n+1}$.
(viii): (0) $V(2)$ says $2<4$, true. (1) Let $n\geq 2$. Assumption: $n!<n^n$. To prove: $(n+1)!<(n+1)^{n+1}$.
Decomposition: $(n+1)!=(n+1)n!\overset{\text{IP}}{<}(n+1)n^n<(n+1)(n+1)^n=(n+1)^{n+1}$.

Solution 2: (i): $f(n)=0$. Weak principle. (0) $n=0$ checks. (1) Let $n\in\mathbb{N}_0$. Assume that $f(n)=0$. Then $f(n+1)=2f(n)\overset{\text{IP}}{=\!=}0$, works for $n+1$.
(ii): $f(n)=n-1$. Weak principle. (0) $n=1$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(n)=n-1$. Then $f(n+1)=f(n)+1\overset{\text{IP}}{=\!=}n-1+1=(n+1)-1$, works for $n+1$.
(iii): $f(n)=\frac{1}{n}$. Weak principle. (0) $n=1$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(n)=\frac{1}{n}$. Then $f(n+1)=f(n)\frac{n}{n+1}\overset{\text{IP}}{=\!=}\frac{1}{n}\frac{n}{n+1}=\frac{1}{n+1}$, works for $n+1$.
(iv): $f(n)=n$. Strong (modified) principle. (0) $n=1$ a $n=2$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(n)=n$ and $f(n-1)=n-1$. Then $f(n+1)=2f(n)-f(n-1)\overset{\text{IP}}{=\!=}2n-(n-1)=n+1$, works for $n+1$.
(v): $f(n)=1$. Strong (modified) principle. (0) $n=1$, $n=2$ a $n=3$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(n)=1$, $f(n-1)=1$ and $f(n-2)=1$. Then $f(n+1)=f(n)+f(n-1)-f(n-2)\overset{\text{IP}}{=\!=}1+1-1=1$, works for $n+1$.
(vi): $f(n)=\begin{cases}1, & n\text{ odd;}\\ 0, & n\text{ even.}\end{cases}$ Strong (modified) principle: (0) For $n=1,2,3$ checks.

(1) Let $n\in\mathbb{N}$, $n\geq 3$. Assume that $f(k)=\begin{cases}1, & k\text{ odd;}\\ 0, & k\text{ even}\end{cases}$ for $k=n-2,n-1,n$.
a) If n is even, then $n-2$ is even while $n-1$ and $n+1$ are odd. Then $f(n+1)=f(n)+f(n-1)-f(n-2)\overset{\text{IP}}{=\!=}0+1-0=1$, works for odd $n+1$.
b) If n is odd, then $n-2$ is odd while $n-1$ and $n+1$ are even. Then $f(n+1)=f(n)+f(n-1)-f(n-2)\overset{\text{IP}}{=\!=}1+0-1=0$, works for even $n+1$.
Alternative: $f(n)=\frac{1}{2}(1-(-1)^n)$, then one can prove directly, using the induction hypothesis one gets

$f(n+1)=f(n)+f(n-1)-f(n-2)\overset{\text{IP}}{=}\frac{1}{2}(1-(-1)^n)+\frac{1}{2}(1-(-1)^{n-1})-\frac{1}{2}(1-(-1)^{n-2})=$
$=\frac{1}{2}(1-(-1)^n+1+(-1)^n-1+(-1)^n)=\frac{1}{2}(1+(-1)^n)=\frac{1}{2}(1-(-1)^{n+1})$.
(vii): $f(n)=3^n$. Strong (modified) principle. (0) $n=1$ and $n=2$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(n)=3^n$ and $f(n-1)=3^{n-1}$. Consider $n+1$.
Je-li $n+1$ sud, pak $f(n+1)=9f(n-1)\overset{\text{IP}}{=}9\cdot 3^{n-1}=3^{n+1}$.
Je-li $n+1$ lich, pak $f(n+1)=3f(n)\overset{\text{IP}}{=}3\cdot 3^n=3^{n+1}$.
(viii): $f(n)=2^{\lfloor n/2\rfloor}$. Strong (modified) principle. (0) $n=1$ and $n=2$ checks. (1) Let $n\in\mathbb{N}$. Assume that $f(k)=\begin{cases}2^{k/2}, & k \text{ even;}\\ 2^{(k-1)/2}, & k \text{ odd}\end{cases}$ for $k=n-1,n$. Consider $n+1$.
If $n+1$ is even, then also $n-1$ is even and $f(n+1)=2f(n-1)\overset{\text{IP}}{=}2\cdot 2^{(n-1)/2}=2^{(n+1)/2}$.
If $n+1$ is odd, then also $n-1$ is odd and $f(n+1)=2f(n-1)\overset{\text{IP}}{=}2\cdot 2^{(n-1-1)/2}=2^{n/2}=2^{((n+1)-1)/2}$.

Solution 3: (i): Strong (modified) induction (0) For $n=1,2$ it checks. (1) Let $n\geq 2$, assuming validity of $V(k)$: $f(n)\leq n!$ and $f(n-1)\leq(n-1)!$. Then $f(n+1)=f(n)+n\,f(n-1)\overset{\text{IP}}{\leq}n!+n\cdot(n-1)!=2n!\leq(n+1)n!=(n+1)!$.
(ii): Strong (modified) induction (0) For $n=1,2$ it checks. (1) Let $n\geq 2$, assuming validity of $V(k)$: $f(n)\leq n^2$ and $f(n-1)\leq(n-1)^2$. Then $f(n+1)=\frac{1}{k}f(n)+f(n-1)\overset{\text{IP}}{\leq}\frac{1}{n}n^2+(n-1)^2=n+n^2-2n+1=n^2-n+1\leq n^2+2n+1=(n+1)^2$.
(iii): Strong (modified) induction (0) For $n=1,2$ it checks. (1) Let $n\geq 2$, assuming validity of $V(k)$: $f(n)=n!$ and $f(n-1)=(n-1)!$. Then $f(n+1)=n\,f(n)+n\,f(n-1)\overset{\text{IP}}{=}n\cdot n!+n\cdot(n-1)!=n\cdot n!+n!=(n+1)\cdot n!=(n+1)!$.
(iv): Strong (modified) induction (0) For $n=1,2$ it checks. (1) Let $n\geq 2$, assuming validity of $V(k)$: $f(n)\geq n!$ and $f(n-1)\leq(n-1)!$. Then $f(n+1)=n\,f(n)+n^2f(n-1)\overset{\text{IP}}{\leq}n\cdot n!+n^2\cdot(n-1)!=n\cdot n!+n\cdot n!=2n\cdot n!\geq(n+1)\cdot n!=(n+1)!$.

Solution 4: (i): (0a) $0\in M$. (0b) $1\in M$. (0c) $10\in M$.
(1a) $w\in M\implies w1\in M$. (1b) $w\in M\implies w10\in M$.
Remark: Without (0c) we can't get 101.
(ii): (0) $0\in M$.
(1a) $w\in M\implies 0w\in M$. (1b) $w\in M\implies 1w\in M$.
Remark: We have to add from the left to guarantee the correct ending on the right.
(iii): (0) $1\in M$.
(1a) $w\in M\implies 0w\in M$. (1b) $w\in M\implies 1w\in M$.
Remark: We have to add from the left to guarantee the correct ending on the right.
(iv): (0) $101\in M$.
(1a) $w\in M\implies 0w\in M$. (1b) $w\in M\implies 1w\in M$. (1c) $w\in M\implies w0\in M$.
(1d) $w\in M\implies w1\in M$.

Solution 5: (i): (0a) $c\in C\implies c\in M$. (0b) $c\in C-\{3\}\implies c3\in M$.
(1a) $[w\in M\wedge c\in C-\{3\}]\implies wc\in M$. (1b) $[w\in M\wedge c\in C-\{3\}]\implies wc3\in M$.
Remark: Without (0b) we can't get 13.
(ii): (0) $2\in M$. (1) $[w\in M\wedge c\in C]\implies wc\in M$.
(iii): (0) $c\in C-\{1\}\implies c\in M$. (1) $[w\in M\wedge c\in C]\implies cw\in M$.
Remark: We have to add from the left to guarantee the correct ending on the right.
(iv): (0a) $\lambda\in M$ (0b) $[c\in\{1,3\}\wedge d\in\{2,4\}]\implies cd\in M$.
(0c) $[c\in\{1,3\}\wedge d\in\{2,4\}]\implies dc\in M$.
(1a) $[r,s\in M\wedge c\in\{1,3\}\wedge d\in\{2,4\}]\implies rcsd\in M$.
(1b) $[r,s\in M\wedge c\in\{1,3\}\wedge d\in\{2,4\}]\implies rdsc\in M$.
Remark: We add from the right, each time we insert a number of opposite parity in the middle. Is that the right way? Proof of correctness of the produced words can be done recursively, we remove the right character and also some character of opposite parity from rest of the chain. We do have to allow for removing from the middle, see the chain 11222211, we do not need to make allowance for taking away from the left end. If a chain has at least 4 characters, then there must be at least two characters of opposite parity, so one of them must be in the middle for taking out.
We had to add (0a), otherwise we could not create chains like 1122.