Calculus 1 Sequences: Solution

Note: The parts between pairs of $\langle\!\langle\ \rangle\!\rangle$ are explanatory notes. These are not "official" parts of the solution, they merely illustrate how we think about the problem. When writing an "official" solution of a problem, such parts should be left out.

1.

$$\lim_{n\to\infty}\Big[\Big(\frac{1-4n^2+3n}{3+n+2n^2}\Big)^3+\frac{1+e^{-n}}{n^2+\frac{1}{n}}\Big]=\Big\langle\!\!\Big\langle\Big(\frac{1-\infty+\infty}{\infty}\Big)^3+\frac{1+e^{-\infty}}{\infty+\frac{1}{\infty}}\Big\rangle\!\!\Big\rangle$$

$$=\Big(\lim_{n\to\infty}\Big[\frac{\frac{1}{n^2}(1-4n^2+3n)}{\frac{1}{n^2}(3+n+2n^2)}\Big]\Big)^3+\lim_{n\to\infty}\Big(\frac{1+\frac{1}{e^n}}{n^2+\frac{1}{n}}\Big)=\Big\langle\!\!\Big\langle ?+\frac{1+\frac{1}{e^\infty}}{\infty+\frac{1}{\infty}}=?+\frac{1+\frac{1}{\infty}}{\infty+0}=?+\frac{1}{\infty}\Big\rangle\!\!\Big\rangle$$

$$=\Big(\lim_{n\to\infty}\Big[\frac{\frac{1}{n^2}-4+\frac{3}{n}}{\frac{3}{n^2}+\frac{1}{n}+2}\Big]\Big)^3+0=\Big\langle\!\!\Big\langle\Big(\frac{\frac{1}{\infty}-4+\frac{3}{\infty}}{\frac{3}{\infty}+\frac{1}{\infty}+2}\Big)^3=\Big(\frac{0-4+0}{0+0+2}\Big)^3\Big\rangle\!\!\Big\rangle=(-2)^3=-8.$$

2.

$$\lim_{n\to\infty}\Big[\cos\Big(\frac{2^{2n+1}+(-2)^n}{5^n+2^{2n-1}}\Big)\Big]=\Big\langle\!\!\Big\langle\cos\Big(\frac{\infty+?}{\infty+\infty}\Big)\Big\rangle\!\!\Big\rangle$$

$$=\cos\Big(\lim_{n\to\infty}\Big[\frac{2\cdot 4^n+(-2)^n}{5^n+\frac{1}{2}\cdot 4^n}\Big]\Big)$$

$$=\Big\langle\!\!\Big\langle n\sim\infty\implies\frac{2\cdot 4^n+(-2)^n}{5^n+\frac{1}{2}\cdot 4^n}\sim\frac{2\cdot 4^n}{5^n}\implies\text{cancel } 4^n \text{ or } 5^n\Big\rangle\!\!\Big\rangle$$

$$=\cos\Big(\lim_{n\to\infty}\Big[\frac{\frac{1}{4^n}\big(2\cdot 4^n+(-2)^n\big)}{\frac{1}{4^n}\big(5^n+\frac{1}{2}\cdot 4^n\big)}\Big]\Big)=\cos\Big(\lim_{n\to\infty}\Big[\frac{2+\big(\frac{-2}{4}\big)^n}{\big(\frac{5}{4}\big)^n+\frac{1}{2}}\Big]\Big)$$

$$=\Big\langle\!\!\Big\langle\big|-\tfrac{1}{2}\big|<1\implies\big(-\tfrac{1}{2}\big)^n\to 0,\quad\big|\tfrac{5}{4}\big|>1\implies\big(\tfrac{5}{4}\big)^n\to\infty,\Big\rangle\!\!\Big\rangle\quad\Big\langle\!\!\Big\langle\frac{2+0}{\infty+\frac{1}{2}}=\frac{2}{\infty}=0\Big\rangle\!\!\Big\rangle$$

$$=\cos(0)=1.$$

3.

$$\lim_{n\to\infty}\big(\sqrt{n^2+4n}-\sqrt{n^2+1}\big)=\Big\langle\!\!\Big\langle\sqrt{\infty}-\sqrt{\infty}\Big\rangle\!\!\Big\rangle$$

$$=\lim_{n\to\infty}\Big(\frac{(\sqrt{n^2+4n}-\sqrt{n^2+1})(\sqrt{n^2+4n}+\sqrt{n^2+1})}{(\sqrt{n^2+4n}+\sqrt{n^2+1})}\Big)=\lim_{n\to\infty}\Big(\frac{n^2+4n-(n^2+1)}{\sqrt{n^2+4n}+\sqrt{n^2+1}}\Big)$$

$$=\lim_{n\to\infty}\Big(\frac{4n-1}{\sqrt{n^2+4n}+\sqrt{n^2+1}}\Big)$$

$$=\Big\langle\!\!\Big\langle n\sim\infty\implies\frac{4n-1}{\sqrt{n^2+4n}+\sqrt{n^2+1}}\sim\frac{4n}{\sqrt{n^2}+\sqrt{n^2}}=\frac{4n}{2n}=2\implies\text{cancel } n\Big\rangle\!\!\Big\rangle$$

$$=\lim_{n\to\infty}\Big(\frac{\frac{1}{n}(4n-1)}{\frac{1}{n}(\sqrt{n^2+4n}+\sqrt{n^2+1})}\Big)=\lim_{n\to\infty}\Big(\frac{4-\frac{1}{n}}{\frac{\sqrt{n^2+4n}}{\sqrt{n^2}}+\frac{\sqrt{n^2+1}}{\sqrt{n^2}}}\Big)$$

$$=\lim_{n\to\infty}\Big(\frac{4-\frac{1}{n}}{\sqrt{\frac{n^2+4n}{n^2}}+\sqrt{\frac{n^2+1}{n^2}}}\Big)=\lim_{n\to\infty}\Big(\frac{4-\frac{1}{n}}{\sqrt{1+\frac{4}{n}}+\sqrt{1+\frac{1}{n^2}}}\Big)$$

$$=\Big\langle\!\!\Big\langle\frac{4-\frac{1}{\infty}}{\sqrt{1+\frac{4}{\infty}}+\sqrt{1+\frac{1}{\infty}}}=\frac{4-0}{\sqrt{1+0}+\sqrt{1+0}}=\frac{4}{1+1}\Big\rangle\!\!\Big\rangle=2.$$

4.

$$\lim_{n\to\infty}\Big(\frac{n-1}{n+1}\Big)^{2n-1} = \Big\langle\!\!\Big\langle \big(\tfrac{\infty}{\infty}\big)^{\infty};\quad \frac{n-1}{n+1}=\frac{1-\frac{1}{n}}{1+\frac{1}{n}} \implies 1^{\infty}=? \Big\rangle\!\!\Big\rangle = \lim_{n\to\infty}\Big(1-\frac{2}{n+1}\Big)^{2n-1}$$

$$= \Big\langle\!\!\Big\langle m=n+1 \implies m\to\infty,\ n=m-1,\ 2n-1=2(m-1)-1=2m-3 \Big\rangle\!\!\Big\rangle$$

$$= \lim_{m\to\infty}\Big(1-\frac{2}{m}\Big)^{2m-3} = \lim_{m\to\infty}\Big[\Big(1-\frac{2}{m}\Big)^{2m}\cdot\Big(1-\frac{2}{m}\Big)^{-3}\Big]$$

$$= \Big[\lim_{m\to\infty}\Big(1+\frac{-2}{m}\Big)^{m}\Big]^{2}\cdot\Big[\lim_{m\to\infty}\Big(1-\frac{2}{m}\Big)\Big]^{-3} = \big[e^{-2}\big]^{2}\cdot(1-0)^{-3} = e^{-4}.$$

5. a) $\{n+(-1)^n\}_{n=1}^{\infty} = \{1-1,2+1,3-1,4+1,5-1,6+1,7-1,\dots\} = \{0,3,2,5,4,7,6,\dots\}$
$\implies \lim\limits_{n\to\infty}\big(n+(-1)^n\big)=\infty.$

Note: $a_n = n+(-1)^n \geq n-1$ and $b_n = n-1\to\infty$.

5. b) $\{n\cdot(-1)^n\}_{n=1}^{\infty} = \{-1,2,-3,4,-5,6,-7,\dots\} \implies \lim\limits_{n\to\infty}\big(n\cdot(-1)^n\big)$ DNE.

5. c)
$\{\sin\big(\frac{\pi}{2}n\big)\}_{n=1}^{\infty} = \{\sin\big(\frac{\pi}{2}\big),\sin(\pi),\sin\big(\frac{3}{2}\pi\big),\sin(2\pi),\sin\big(\frac{5}{2}\pi\big),\sin(3\pi),\sin\big(\frac{7}{2}\pi\big),\sin(4\pi),\dots\} =$
$\{1,0,-1,0,1,0,-1,0,\dots\} \implies \lim\limits_{n\to\infty}\Big(\sin\big(\frac{\pi}{2}n\big)\Big)$ DNE.

5. d)
$\{\frac{1}{n}\sin\big(\frac{\pi}{2}n\big)\}_{n=1}^{\infty} = \{\sin\big(\frac{\pi}{2}\big),\frac{1}{2}\sin(\pi),\frac{1}{3}\sin\big(\frac{3}{2}\pi\big),\frac{1}{4}\sin(2\pi),\frac{1}{5}\sin\big(\frac{5}{2}\pi\big),\frac{1}{6}\sin(3\pi),\frac{1}{7}\sin\big(\frac{7}{2}\pi\big),\dots\} =$
$\{1,0,-\frac{1}{3},0,\frac{1}{5},0,-\frac{1}{7},0,\dots\} \implies \lim\limits_{n\to\infty}\Big(\frac{1}{n}\sin\big(\frac{\pi}{2}n\big)\Big)=0.$