Calculus 1 Function: Solution

Note: The parts between pairs of $\langle\!\langle\ \rangle\!\rangle$ are explanatory notes. These are not "official" parts of the solution, they merely illustrate how we think about the problem. When writing an "official" solution of a problem, such parts should be left out.

1a.

— Clearly $x \neq 0$.
— $\arcsin(y)$ requires $-1 \le y \le 1 \implies -1 \le \frac{x+1}{x} \le 1$. One way to solve: multiply by x, careful about sign! If $x > 0$, obtain $-x \le x+1 \le x$; these two equations have no solution (right equation reads $1 \le 0$). If $x < 0$, obtain $-x \ge x+1 \ge x$; these two equations yield $x \in \left(-\infty, -\frac{1}{2}\right]$.
— $\sqrt{y}$ requires $y \ge 0 \implies \ln(2x+7) \ge 0 \implies 2x+7 \ge 1 \implies x \in [-3,\infty)$.
— $\ln(y)$ requires $y > 0 \implies 2x+7 > 0$, but the previous condition is stronger $\implies$ we already took care of this.
Putting these together we get $D_f = \left[-3, -\frac{1}{2}\right]$.

1b.

— $\sqrt{y}$ requires $y \ge 0 \implies \cos(x) \ge 0 \implies x \in \bigcup\limits_{k=-\infty}^{\infty} \left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right]$.
— $\sqrt[x]{\frac{1+x}{2-x}}$ must be written as $e^{\frac{1}{x}\ln\left(\frac{1+x}{2-x}\right)}$. Clearly $x \neq 0$. $\ln(y)$ requires $y > 0 \implies \frac{1+x}{2-x} > 0$. Favourite method (for instance, checking on signs in intervals $-\infty$ to -1, -1 to 2, 2 to ∞) gives that $-1 < x < 2$, but the zero is out $\implies x \in (-1,0) \cup (0,2)$.
Putting the two conditions together we get (using $\pi/2 \sim 1.7$) that $D_f = (-1,0) \cup \left(0, \frac{\pi}{2}\right]$.

2.

— First, $\ln(y)$ requires $y > 0 \implies 1 - e^{2x} > 0 \implies e^{2x} < 1 \implies 2x < 0 \implies D_f = (-\infty, 0) = I\!R^-$.
— Inverse: $y = \ln(1-e^{2x}) \implies e^y = 1 - e^{2x} \implies e^{2x} = 1 - e^y \implies 2x = \ln(1-e^y) \implies x = f^{-1}(y) = \frac{1}{2}\ln(1-e^y)$.
— We know that $\operatorname{ran}(f^{-1}) = D_f$, so $\operatorname{ran}(f^{-1}) = (-\infty, 0)$.
— If $y_1 < y_2$, then (since exp is increasing) $e^{y_1} < e^{y_2} \implies 1 - e^{y_1} > 1 - e^{y_2} \implies$ (since ln is increasing) $\ln(1-e^{y_1}) > \ln(1-e^{y_2}) \implies f^{-1}(y_1) > f^{-1}(y_2)$. Thus f^{-1} is decreasing (and hence monotone), the above resoning is already a proof.

3.

a)

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{\sqrt{2x-1}-1}{x-1}\right) &= \left\langle\!\!\left\langle \frac{0}{0} \implies \text{trick for } \sqrt{\ } - \sqrt{\ } \right\rangle\!\!\right\rangle \\
&= \lim_{x\to 1}\left(\frac{\left(\sqrt{2x-1}-1\right)\left(\sqrt{2x-1}+1\right)}{(x-1)\left(\sqrt{2x-1}+1\right)}\right) = \lim_{x\to 1}\left(\frac{2x-1-1^2}{(x-1)\left(\sqrt{2x-1}+1\right)}\right) \\
&= \lim_{x\to 1}\left(\frac{2(x-1)}{(x-1)\left(\sqrt{2x-1}+1\right)}\right) = \lim_{x\to 1}\left(\frac{2}{\sqrt{2x-1}+1}\right) \\
&= \frac{2}{1+1} = 1.
\end{aligned}$$

b) $\frac{x+1}{x-1} = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} \to 1$ as $x \to \infty$ and $\lim\limits_{x\to\infty}\big(\arctan(x)\big) = \frac{\pi}{2}$, hence

$$\lim_{x\to\infty}\left[\ln\left(\frac{x+1}{x-1}\right) + \sin\big(\arctan(x)\big)\right] = [\ln(1) + \sin(\pi/2)] = 0 + 1 = 1.$$

c) & d): Type is common, $\infty - \infty$ or even worse (we divide by some kind of zero). The trick is to somehow make the expression into one, here the common denominator is really inviting:

$$\frac{1}{x}-\frac{1}{x^3}=\frac{x^2-1}{x^3}.$$

c) If $x\to 0^-$, then $x^3\to 0^-$, hence

$$\lim_{x\to 0^-}\Big(\frac{1}{x}-\frac{1}{x^3}\Big)\lim_{x\to 0^-}\Big(\frac{x^2-1}{x^3}\Big)=\Big\langle\!\!\Big\langle\frac{-1}{0^-}=-\frac{1}{0^-}=-(-\infty)\Big\rangle\!\!\Big\rangle=\infty.$$

d) If $x\to 0^+$, then $x^3\to 0^+$, hence

$$\lim_{x\to 0^+}\Big(\frac{1}{x}-\frac{1}{x^3}\Big)\lim_{x\to 0^+}\Big(\frac{x^2-1}{x^3}\Big)=\Big\langle\!\!\Big\langle\frac{-1}{0^+}=-\frac{1}{0^+}=-\infty\Big\rangle\!\!\Big\rangle=-\infty.$$

From c) we have

$$\lim_{x\to 0^-}\Big(\frac{1}{x}-\frac{1}{x^3}\Big)=\infty.$$

Since limits from the left and right do not agree, $\lim\limits_{x\to 0}\big(\frac{1}{x}-\frac{1}{x^3}\big)$ does not exist (DNE).

e).

$$\lim_{x\to 0^+}\Big(\frac{\arcsin(\sqrt{x})}{\sqrt{x-x^2}}\Big)=\Big\langle\!\!\Big\langle\frac{0}{0}\implies \text{LH}\Big\rangle\!\!\Big\rangle=\lim_{x\to 0^+}\Big(\frac{[\arcsin(\sqrt{x})]'}{[\sqrt{x-x^2}]'}\Big)=\lim_{x\to 0^+}\Big(\frac{\frac{1}{\sqrt{1-\sqrt{x}^2}}[\sqrt{x}]'}{\frac{1}{2\sqrt{x-x^2}}[x-x^2]'}\Big)$$

$$=\lim_{x\to 0^+}\Big(\frac{\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x-x^2}}(1-2x^2)}\Big)=\lim_{x\to 0^+}\Big(\frac{\sqrt{x-x^2}}{\sqrt{x}\sqrt{1-x}(1-2x^2)}\Big)=\Big\langle\!\!\Big\langle\frac{0}{0}\implies \text{try algebra}\Big\rangle\!\!\Big\rangle$$

$$=\lim_{x\to 0^+}\Big(\frac{\sqrt{x-x^2}}{\sqrt{x-x^2}(1-2x^2)}\Big)=\lim_{x\to 0^+}\Big(\frac{1}{1-2x^2}\Big)=1.$$

f).

$$\lim_{x\to\infty}\Big(\frac{e^{\sqrt{x}}}{x-2}\Big)=\Big\langle\!\!\Big\langle\frac{\infty}{\infty}\implies \text{LH}\Big\rangle\!\!\Big\rangle=\lim_{x\to\infty}\Big(\frac{[e^{\sqrt{x}}]'}{[x-2]'}\Big)=\lim_{x\to\infty}\Big(\frac{e^{\sqrt{x}}[\sqrt{x}]'}{1}\Big)=\lim_{x\to\infty}\Big(\frac{e^{\sqrt{x}}}{2\sqrt{x}}\Big)$$

$$=\Big\langle\!\!\Big\langle\frac{\infty}{\infty}\implies \text{LH}\Big\rangle\!\!\Big\rangle=\lim_{x\to\infty}\Big(\frac{[e^{\sqrt{x}}]'}{[2\sqrt{x}]'}\Big)=\lim_{x\to\infty}\Big(\frac{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}{\frac{2}{2\sqrt{x}}}\Big)=\lim_{x\to\infty}\big(\tfrac{1}{2}e^{\sqrt{x}}\big)=\infty.$$

g) The general power has to be handled first, then we try to substitute in:

$$\lim_{x\to 0^+}\Big[\Big(\frac{1}{x}\Big)^{\sin(x)}\Big]=\lim_{x\to 0^+}\Big[e^{\sin(x)\ln\left(\frac{1}{x}\right)}\Big]=e^{0\cdot\ln(1/0^+)}=e^{0\cdot\ln(\infty)}=e^{0\cdot\infty}.$$

This is a problem, so we have to use our standard trick for $0\cdot\infty$. One part has to be moved into the denominator, we will try sine. Why? Because then logarithm stays all by itself, hence it will be killed by the derivative. The sine will get more complicated by the derivative, but hopefully it will work out (after all, it is a homework problem, so there must be a way out). To simplify our work we first do the limit of exponent only.

$$\lim_{x\to 0^+}\Big(\sin(x)\ln\big(\tfrac{1}{x}\big)\Big)=\lim_{x\to 0^+}\Big(\frac{-\ln(x)}{\sin(x)^{-1}}\Big)=\Big\langle\!\!\Big\langle\frac{\infty}{\infty}\implies \text{LH}\Big\rangle\!\!\Big\rangle=\lim_{x\to 0^+}\Big(\frac{[-\ln(x)]'}{[\sin(x)^{-1}]'}\Big)$$

$$=\lim_{x\to 0^+}\Big(\frac{-\frac{1}{x}}{-\sin(x)^{-2}\cos(x)}\Big)=\lim_{x\to 0^+}\Big(\frac{\sin^2(x)}{x\cos(x)}\Big)$$

$$=\Big\langle\!\!\Big\langle\frac{0}{0}\implies \text{LH}\Big\rangle\!\!\Big\rangle=\lim_{x\to 0^+}\Big(\frac{2\sin(x)\cos(x)}{\cos(x)-x\sin(x)}\Big)=\frac{0}{1}=0.$$

The last troublesome limit can be also done like this:

$$\lim_{x\to 0^+}\Big(\frac{\sin^2(x)}{x\cos(x)}\Big)=\lim_{x\to 0^+}\Big(\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{\cos(x)}\Big)=1\cdot 0=0.$$

In any case, we have to remember to put back that exponential, so the answer is $e^0=1$.

4. First write the general power properly:

$$f(x) = e^{\ln[\arccos(x)^{1/(x+1)}]} = e^{\dfrac{\ln[\arccos(x)]}{x+1}}.$$

Domain: $x \neq -1$, arccos requires $x \in [-1,1]$, and logarithm needs $\arccos(x) > 0 \implies x \neq 1$. Thus $D_f = (-1,1)$.
Limits (only one-sided possible!):

$$\lim_{x\to 1^-}[f(x)] = e^{\lim\limits_{x\to 1^-}\left(\frac{\ln[\arccos(x)]}{x+1}\right)} = e^{\dfrac{\ln(0^+)}{2}} = e^{\dfrac{-\infty}{2}} = e^{-\infty} = 0.$$

$$\lim_{x\to -1^+}[f(x)] = e^{\lim\limits_{x\to -1^+}\left(\frac{\ln[\arccos(x)]}{x+1}\right)} = e^{\dfrac{\ln(\pi)}{0^+}} = e^{\ln(\pi)\dfrac{1}{0^+}} = e^{\ln(\pi)\cdot\infty}$$
$$= \left\langle\!\!\left\langle \pi > 1 \implies \ln(\pi) > 0 \right\rangle\!\!\right\rangle = e^{\infty} = \infty.$$

5. All three functions are continuous on their respective domains, therefore $f(x)$ is also continuous on open intervals where it is defined by individual functions. Consequently, f is continuous on $(-\infty,0)$, on $(0,1)$, and on $(1,\infty)$. To find continuity at 0 and 1, check on one-sided limits and compare to f:
— $f(0) = 1 - 0 = 1$;

$$\lim_{x\to 0^+}[f(x)] = \left\langle\!\!\left\langle x \to 0^+ \implies x > 0,\ x < 1 \right\rangle\!\!\right\rangle = \lim_{x\to 0^+}(1-x) = 1;$$

$$\lim_{x\to 0^-}[f(x)] = \left\langle\!\!\left\langle x \to 0^- \implies x < 0 \right\rangle\!\!\right\rangle = \lim_{x\to 0^-}\left(\frac{\sin(3x)}{x}\right)$$
$$= \left\langle\!\!\left\langle y = 3x \implies x = \frac{1}{3}y,\ y \to 0^- \right\rangle\!\!\right\rangle = \lim_{y\to 0^-}\left(3\frac{\sin(y)}{y}\right) = 3\cdot 1 = 3.$$

Since one-sided limits are not equal, the limit of f at 0 does not exist, hence f is not continuous at 0. But one-sided limits do exist finite, which means that we have a jump discontinuity. Note f is also continuous at 0 from the right.
— $f(1) = 1 - 1 = 0$;

$$\lim_{x\to 1^+}[f(x)] = \left\langle\!\!\left\langle x \to 1^+ \implies x > 1 \right\rangle\!\!\right\rangle = \lim_{x\to 1^+}\left(e^{1/(1-x)}\right) = e^{1/0^-} = e^{-\infty} = 0.$$

$$\lim_{x\to 1^-}[f(x)] = \left\langle\!\!\left\langle x \to 1^- \implies x < 1,\ x > 0 \right\rangle\!\!\right\rangle = \lim_{x\to 1^-}(1-x) = 0;$$

Since one-sided limits exist finite and are equal, the limit of f at 1 also exists; it is equal to $f(1)$, hence f is continuous at 1.
Conclusion: f is continuous on $(-\infty,0)\cup(0,\infty)$, there is a jump discontinuity at 0.