Calculus 1 Derivative: Solution

Note: The parts between pairs of $\langle\!\langle\ \rangle\!\rangle$ are explanatory notes. These are not "official" parts of the solution, they merely illustrate how we think about the problem. When writing an "official" solution of a problem, such parts should be left out.

1a. Domain: 2^{-x} accepts all real x, arcsin requires that $-1 \le 2x - \sqrt{x} \le 1$. This can be solved using substitution $\sqrt{x} = y$. Thus $-1 \le 2x - \sqrt{x}$ becomes $2y^2 - y + 1 \ge 0$, which is always true $\implies$ no condition. $2x - \sqrt{x} \le 1$ becomes $2y^2 - y - 1 \le 0$. Zero points are (from formula) $-\frac{1}{2}$, 1. Thus we need $-\frac{1}{2} \le \sqrt{x} \le 1$. Since square root is never negative, this becomes $0 \le \sqrt{x} \le 1 \implies x \in [0,1]$.
Finally, in the denominator we need arcsin $\neq 0$, that is, $2x - \sqrt{x} \neq 0 \implies x \neq 0$ and $x \neq \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Conclusion: $D_f = \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, 1\right]$.
Derivative:

$$f'(x) = \left[\frac{2^{-x}}{\arcsin(2x-\sqrt{x})}\right]' = \left\langle\!\!\left\langle \text{last: fraction} \implies \text{quotient rule} \right\rangle\!\!\right\rangle$$

$$= \frac{[2^{-x}]' \cdot \arcsin(2x-\sqrt{x}) - 2^{-x}[\arcsin(2x-\sqrt{x})]'}{[\arcsin(2x-\sqrt{x})]^2}$$

$$= \left\langle\!\!\left\langle \text{last: composition} \implies \text{chain rule, } [a^y]' = \ln(a)a^y,\ [\arcsin(y)]' = \frac{1}{\sqrt{1-y^2}} \right\rangle\!\!\right\rangle$$

$$= \frac{\ln(2)2^{-x}[-x]' \cdot \arcsin(2x-\sqrt{x}) - 2^{-x}\frac{1}{\sqrt{1-(2x-\sqrt{x})^2}}[2x-\sqrt{x}]'}{[\arcsin(2x-\sqrt{x})]^2}$$

$$= \left\langle\!\!\left\langle \text{last: polynomials, } [\sqrt{x}]' = \left[x^{1/2}\right]' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \right\rangle\!\!\right\rangle$$

$$= \frac{\ln(2)2^{-x}(-1) \cdot \arcsin(2x-\sqrt{x}) - 2^{-x}\frac{1}{\sqrt{1-4x^2-x+4x\sqrt{x}}}\left(2 - \frac{1}{2\sqrt{x}}\right)}{[\arcsin(2x-\sqrt{x})]^2}$$

$$= \frac{-\ln(2)2^{-x}\arcsin(2x-\sqrt{x}) - \frac{2^{-x}}{\sqrt{1-4x^2-x+4x\sqrt{x}}}\left(2 - \frac{1}{2\sqrt{x}}\right)}{[\arcsin(2x-\sqrt{x})]^2}.$$

1b. Domain: Fraction requires $x \neq -1$.
ln requires $\frac{|x-1|}{x+1} > 0$. Since $|x-1| \ge 0$ always, we need that $x \neq 1$ (to exclude 0) and also $(x+1) > 0$, that is, $x > -1$. Conclusion: $D_f = (-1,1) \cup (1,\infty)$.
To work with the function we eliminate the absolute value, breaking point is when $x-1 = 0$, i.e. $x = 1$:

$$f(x) = \begin{cases} \ln\left(\frac{x-1}{x+1}\right); & x > 1 \\ \ln\left(\frac{-(x-1)}{x+1}\right); & -1 < x < 1 \end{cases}.$$

Thus for $x > 1$:

$$f'(x) = \left[\ln\left(\frac{x-1}{x+1}\right)\right]' = \left\langle\!\!\left\langle \text{last: composition with log} \implies \text{chain rule, } [\ln(y)]' = \frac{1}{y} \right\rangle\!\!\right\rangle$$

$$= \frac{1}{\frac{x-1}{x+1}}\left[\frac{x-1}{x+1}\right]' = \left\langle\!\!\left\langle \text{last: fraction} \implies \text{fraction rule} \right\rangle\!\!\right\rangle$$

$$= \frac{x+1}{x-1}\frac{[x-1]' \cdot (x+1) - (x-1) \cdot [x+1]'}{(x+1)^2} = \frac{x+1}{x-1}\frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2}$$

$$= \frac{x+1}{x-1}\frac{2}{(x+1)^2} = \frac{2}{(x+1)(x-1)}.$$

Note the trick:

$$\left[\frac{x-1}{x+1}\right]' = \left[1 - \frac{2}{x+1}\right]' = \left[1 - 2 \cdot (x+1)^{-1}\right]' = 0 - 2 \cdot (-1)(x+1)^{-2} \cdot [x+1]' = \frac{2}{(x+1)^2}.$$

Similarly for $x<1$ (and $x>-1$):

$$f'(x)=\Big[\ln\Big(-\frac{x-1}{x+1}\Big)\Big]'=\frac{1}{-\frac{x-1}{x+1}}\Big[-\frac{x-1}{x+1}\Big]'$$
$$=-\frac{x+1}{x-1}\cdot(-1)\frac{[x-1]'\cdot(x+1)-(x-1)\cdot[x+1]'}{(x+1)^2}=\frac{2}{(x+1)(x-1)}.$$

Thus on D_f one has

$$f'(x)=\frac{2}{(x+1)(x-1)}.$$

The domain of f' must be a part of D_f, on the other hand we had no trouble differentiating at any point there, so $D_{f'}=D_f=(-1,1)\cup(1,\infty)$.

Note: It would be a mistake to just investigate f' itself for domain, which would yield the incorrect answer $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. A derivative cannot exist where f does not.

1c.

$$f'(x)=\Big\langle\!\Big\langle\text{last: power}\implies\text{chain rule}\Big\rangle\!\Big\rangle=13\Big(\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big)^{12}\cdot\Big[\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big]'$$
$$=\Big\langle\!\Big\langle\text{last: fraction}\implies\text{ratio rule}\Big\rangle\!\Big\rangle$$
$$=13\Big(\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big)^{12}\cdot\frac{[\sin(\sqrt[3]{x})]'e^{1-2x}\ln(x^2+1)-\sin(\sqrt[3]{x})[e^{1-2x}\ln(x^2+1)]'}{e^{1-2x}\ln(x^2+1)}$$
$$=\Big\langle\!\Big\langle\text{left: last: sine}\implies\text{chain rule; right: last: product}\implies\text{product rule}\Big\rangle\!\Big\rangle$$
$$=13\Big(\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big)^{12}\times$$
$$\times\frac{\cos(\sqrt[3]{x})[x^{1/3}]'e^{1-2x}\ln(x^2+1)-\sin(\sqrt[3]{x})\big([e^{1-2x}]'\ln(x^2+1)+e^{1-2x}[\ln(x^2+1)]'\big)}{[e^{1-2x}\ln(x^2+1)]^2}$$
$$=\Big\langle\!\Big\langle\text{last: exp and log}\implies\text{chain rule}\Big\rangle\!\Big\rangle=13\Big(\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big)^{12}\times$$
$$\times\frac{\cos(\sqrt[3]{x})\frac{1}{3}x^{-2/3}e^{1-2x}\ln(x^2+1)-\sin(\sqrt[3]{x})\big(e^{1-2x}[1-2x]'\ln(x^2+1)+e^{1-2x}\frac{1}{x^2+1}[x^2+1]'\big)}{[e^{1-2x}]^2\ln^2(x^2+1)}$$
$$=13\Big(\frac{\sin(\sqrt[3]{x})}{e^{1-2x}\ln(x^2+1)}\Big)^{12}\times$$
$$\times\frac{\cos(\sqrt[3]{x})\frac{1}{3}x^{-2/3}e^{1-2x}\ln(x^2+1)-\sin(\sqrt[3]{x})\big(-2e^{1-2x}\ln(x^2+1)+e^{1-2x}\frac{2x}{x^2+1}\big)}{[e^{1-2x}]^2\ln^2(x^2+1)}$$
$$=13\sin^{12}(\sqrt[3]{x})\frac{\cos(\sqrt[3]{x})\frac{1}{3}x^{-2/3}\ln(x^2+1)-\sin(\sqrt[3]{x})\big(-2\ln(x^2+1)+\frac{2x}{x^2+1}\big)}{[e^{1-2x}]^{13}\ln^{14}(x^2+1)}.$$

1d. First we have to rewrite the general power, then differentiate:

$$f'(x) = \Big[\big(2-\sin(x)\big)^{1/\cos(x)}\Big]' = \Big[e^{\frac{\ln(2-\sin(x))}{\cos(x)}}\Big]' = \Big\langle\!\!\Big\langle \text{last: exp} \implies \text{chain rule}\Big\rangle\!\!\Big\rangle$$

$$= e^{\frac{\ln(2-\sin(x))}{\cos(x)}} \cdot \Big[\frac{\ln\big(2-\sin(x)\big)}{\cos(x)}\Big]' = \Big\langle\!\!\Big\langle \text{last: ratio} \implies \text{ratio rule}\Big\rangle\!\!\Big\rangle$$

$$= e^{\frac{\ln(2-\sin(x))}{\cos(x)}} \cdot \Big(\frac{[\ln\big(2-\sin(x)\big)]'\cos(x) - \ln\big(2-\sin(x)\big)[\cos(x)]'}{[\cos(x)]^2}\Big)$$

$$= \Big\langle\!\!\Big\langle \text{last: log} \implies \text{chain rule}\Big\rangle\!\!\Big\rangle$$

$$= e^{\frac{\ln(2-\sin(x))}{\cos(x)}} \cdot \frac{\frac{1}{2-\sin(x)}[2-\sin(x)]'\cos(x) - \ln\big(2-\sin(x)\big)\big(-\sin(x)\big)}{\cos^2(x)}$$

$$= \sqrt[\cos(x)]{2-\sin(x)} \cdot \frac{\frac{-\cos(x)}{2-\sin(x)}\cos(x) + \sin(x)\ln\big(2-\sin(x)\big)}{\cos^2(x)}$$

$$= \sqrt[\cos(x)]{2-\sin(x)} \cdot \Big(\frac{\sin(x)\ln\big(2-\sin(x)\big)}{\cos^2(x)} - \frac{1}{2-\sin(x)}\Big).$$

2. To find the lines, we need to identify the point and slopes. The point is easy to find, $x_0 = 1$ is given and $y_0 = f(1) = 1^1 = 1$. To find the slopes we need to find the derivative. First we have to write the general power properly:

$$f(x) = \Big(\frac{2x^2}{x^2+1}\Big)^x = e^{\ln\left[\left(\frac{2x^2}{x^2+1}\right)^x\right]} = e^{x\ln\left(\frac{2x^2}{x^2+1}\right)}.$$

Derivative:

$$f'(x) = \Big[e^{x\ln\left(\frac{2x^2}{x^2+1}\right)}\Big]'$$

$$= \Big\langle\!\!\Big\langle \text{last: composition with exponential} \implies \text{chain rule, } [e^y]' = e^y\Big\rangle\!\!\Big\rangle$$

$$= e^{x\ln\left(\frac{2x^2}{x^2+1}\right)} \cdot \Big[x\ln\Big(\frac{2x^2}{x^2+1}\Big)\Big]' = \Big\langle\!\!\Big\langle \text{last: product} \implies \text{product rule}\Big\rangle\!\!\Big\rangle$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big([x]' \cdot \ln\Big(\frac{2x^2}{x^2+1}\Big) + x \cdot \Big[\ln\Big(\frac{2x^2}{x^2+1}\Big)\Big]'\Big)$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(1 \cdot \ln\Big(\frac{2x^2}{x^2+1}\Big) + x \cdot \Big[\ln\Big(\frac{2x^2}{x^2+1}\Big)\Big]'\Big)$$

$$= \Big\langle\!\!\Big\langle \text{last: composition with log} \implies \text{chain rule, } [\ln(y)]' = \tfrac{1}{y}\Big\rangle\!\!\Big\rangle$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(\ln\Big(\frac{2x^2}{x^2+1}\Big) + x \cdot \frac{1}{\frac{2x^2}{x^2+1}} \cdot \Big[\frac{2x^2}{x^2+1}\Big]'\Big)$$

$$= \Big\langle\!\!\Big\langle \text{last: fraction} \implies \text{fraction rule}\Big\rangle\!\!\Big\rangle$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(\ln\Big(\frac{2x^2}{x^2+1}\Big) + x \cdot \frac{x^2+1}{2x^2} \cdot \frac{[2x^2]' \cdot (x^2+1) - 2x^2 \cdot [x^2+1]'}{(x^2+1)^2}\Big)$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(\ln\Big(\frac{2x^2}{x^2+1}\Big) + \frac{x^2+1}{2x} \cdot \frac{4x \cdot (x^2+1) - 2x^2 \cdot 2x}{(x^2+1)^2}\Big)$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(\ln\Big(\frac{2x^2}{x^2+1}\Big) + \frac{x^2+1}{2x} \cdot \frac{4x}{(x^2+1)^2}\Big)$$

$$= \Big(\frac{2x^2}{x^2+1}\Big)^x \cdot \Big(\ln\Big(\frac{2x^2}{x^2+1}\Big) + \frac{2}{x^2+1}\Big).$$

Note: One can also use the following trick using algebra and the chain rule:

$$\Big[\frac{2x^2}{x^2+1}\Big]' = \Big[\frac{2x^2+2-2}{x^2+1}\Big]' = \Big[2-\frac{2}{x^2+1}\Big]'$$
$$= \big[2-2(x^2+1)^{-1}\big]' = 0-2\cdot(-1)(x^2+1)^{-2}\cdot[x^2+1]' = \frac{4x}{(x^2+1)^2}.$$

Thus we can find the slope of the tangent line: $k_T = f'(1) = 1\cdot\big(\ln(1)+1\big) = 1$. Hence the line is given by the equation

$$(y-y_0) = k\cdot(x-x_0) \implies (y-1) = (x-1) \implies y = x.$$

The slope of the normal satisfies $k_T\cdot k_N = -1$, so $k_N = \frac{-1}{k_T} = -1.$. Therefore we get the equation

$$(y-1) = -(x-1) \implies y = 2-x.$$

3. a) It is probably difficult to show directly that the function is 1–1. Instead we will try to show that the function is strictly monotone. Check out the derivative: $f'(x) = 6x^2+6$. Clearly always $f'(x)\geq 6$, so the function is strictly increasing and therefore 1–1. Thus it has an inverse.
b) Since the inverse is hard to find (solving $2x^3+6x+1 = y$ for x is probably impossible), we use formula:

$$[f^{-1}]'(9) = \frac{1}{f'\big(f^{-1}(9)\big)} = \frac{1}{[6x^2+6]\Big|_{x=f^{-1}(9)}}.$$

We need to find $f^{-1}(9)$, in other words, we need to guess x so that $2x^3+6x+1 = 1$. This is easy, after some trying you get $x = 1$, so

$$[f^{-1}]'(9) = \frac{1}{6+6} = \frac{1}{12}.$$

4. $f'(x) = -2x^{-3}$, $f'(1) = -2$, so the total differential is $\mathrm{d}f(1)[h] = -2h$, or $\mathrm{d}f(1) = -2\mathrm{d}x$.
Now:

$f(x) = x^{-2}$	$f(1) = 1$
$f'(x) = -2\,x^{-3}$	$f'(1) = -2 = -2!$
$f''(x) = 2\cdot 3\,x^{-4}$	$f''(1) = 3!$
$f'''(x) = -2\cdot 3\cdot 4\,x^{-5}$	$f'''(1) = -4!$
$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5\,x^{-6}$	$f^{(4)}(1) = 5!$

Thus we have:

$$T_2(x) = f(1)+f'(1)\cdot(x-1)+\frac{1}{2!}f''(1)\cdot(x-1)^2 = 1-2(x-1)+\frac{3!}{2!}(x-1)^2$$
$$= 1-2(x-1)+3(x-1)^2.$$
$$T_3(x) = f(1)+f'(1)\cdot(x-1)+\frac{1}{2!}f''(1)\cdot(x-1)^2+\frac{1}{3!}f'''(1)\cdot(x-1)^3$$
$$= 1-2(x-1)+3(x-1)^2-4(x-1)^3.$$
$$T_4(x) = f(1)+f'(1)\cdot(x-1)+\frac{1}{2!}f''(1)\cdot(x-1)^2+\frac{1}{3!}f'''(1)\cdot(x-1)^3+\frac{1}{4!}f^{(4)}(1)\cdot(x-1)^4$$
$$= 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4.$$

We guess that

$$T_n(x) = 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-\ldots+(-1)^n(n+1)(x-1)^n.$$

We can also try it by the definition. We first guess that $f^{(k)}(x) = (-1)^k (k+1)! x^{-k-2}$ (check that it fits for $k = 0, 1, 2, 3, 4$), so $f^{(k)}(1) = (-1)^k (k+1)!$ and therefore

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} \frac{(-1)^k (k+1)!}{k!}(x-1)^k = \sum_{k=0}^{n}(-1)^k (k+1)(x-1)^k.$$

5. The closest "nice" number to 8 is 9 (we know how to calculate $\sqrt{9} = 3$). So we will find T_2 at $a = 9$ for the function $f(x) = \sqrt{x}$. First:

$f(x) = x^{1/2}$ $f(9) = 9^{1/2} = 3$
$f'(x) = \frac{1}{2} x^{-1/2}$ $f'(9) = \frac{1}{2\cdot 3} = \frac{1}{6}$
$f''(x) = -\frac{1}{4} x^{-3/2}$ $f''(9) = -\frac{1}{4\cdot 3^3} = -\frac{1}{108}$

Thus

$$T_2(x) = f(9) + f'(9)\cdot(x-9) + \frac{f''(9)}{2!}\cdot(x-9)^2 = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2.$$

Substitute: $T_2(8) = 3 + \frac{1}{6}(8-9) - \frac{1}{216}(8-9)^2 = 2 + \frac{179}{216}$. Since we hope that the Taylor polynomial approximates f well close to $a = 9$, we also hope that $\sqrt{8} = f(8) \sim T_2(8)$. Thus we get our approximation $\sqrt{8} \sim 2 + \frac{179}{216} = 2.8287...$

By the way, the precise answer is $\sqrt{8} = 2.8284...$, so the estimate was quite good.