Calculus 1 Shape of functions: Solution

Note: The parts between pairs of slashes $\langle\!\langle \cdots \rangle\!\rangle$ are explanatory notes. These are not "official" parts of the solution, they merely illustrate how we think about the problem. When writing an "official" solution of a problem, such parts should be left out.

1a. First we need to find the domain. Here it is easy, $D_f = (-\infty, 1) \cup (1, \infty)$. To find the intervals of monotonicity, we need the derivative:

$$f'(x) = \Big[\frac{x^2}{x-1}\Big]' = \frac{2x(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Suspicious points: $x = 1$ is a hole in the domain. $f'(x) = 0 \implies x = 0,\ x = 2$. Check on signs of the derivative in intervals given by suspicious points:

	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

From the picture we also see that there is a local maximum at $x = 0$ and a local minimum at $x = 2$. Precisely, $f(0) = 0$ is a local maximum, $f(2) = 4$ is a local minimum.

1b. The function is defined on the whole real line. To get the intervals, we need the derivative of f, and to get this we first need to get rid of the absolute value:

$$f(x) = \begin{cases} x\,e^{-x}; & x \geq 0 \\ x\,e^{-(-x)}; & x \leq 0 \end{cases} = \begin{cases} x\,e^{-x}; & x \geq 0 \\ x\,e^{x}; & x \leq 0 \end{cases}.$$

Thus

$$f'(x) = \begin{cases} e^{-x} - x\,e^{-x}; & x > 0 \\ e^{x} + x\,e^{x}; & x < 0 \end{cases}.$$

Note that we did not get the derivative at 0 in this way because none of the two formulas for f above is valid on a neighborhood of 0. Thus $x = 0$ is a suspicious point. Now we need to solve $f'(x) = 0$ (we will use the fact that $e^y = 0$ has no solution):

$$f'(x) = 0 \iff \begin{cases} (1-x)e^{-x} = 0; & x > 0 \\ (1+x)\,e^{x} = 0; & x < 0 \end{cases} \iff \begin{cases} x = 1; & x > 0 \\ x = -1; & x < 0 \end{cases}.$$

Since both roots belong to the regions where the appropriate equation was relevant (for instance, $x = 1$ was obtained from $(1-x)e^{-x} = 0$, which was only relevant at $x > 0$), we have to consider them. We have three suspicious points and now we check on the signs of f':

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$+$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

From this we see that there is a local minimum $f(-1) = -\frac{1}{e}$ and a local maximum $f(1) = \frac{1}{e}$.

2. We need to check on the values of f at all suspicious points. Since we are looking for global extrema on a closed interval, the endpoints are candidates:
— $f(0) = 3$, $f(4) = 5$.
Other candidates come from suspicious points obtained from the derivative. Again, we need to get rid of absolute value first:

$$f(x) = \begin{cases} (x-3)\cdot(x+1); & x - 3 \geq 0 \\ -(x-3)\cdot(x+1); & x - 3 \leq 0 \end{cases} = \begin{cases} x^2 - 2x - 3; & x \geq 3 \\ -x^2 + 2x + 3; & x \leq 3 \end{cases}.$$

Thus

$$f'(x) = \begin{cases} 2x - 2; & x > 3 \\ 2 - 2x; & x < 3 \end{cases}.$$

Note that we did not get the derivative at 3 in this way again. Thus $x = 3$ is a suspicious point. Now we need to solve $f'(x) = 0$:

$$f'(x) = 0 \iff \begin{cases} 2x - 2 = 0; & x > 3 \\ 2 - 2x = 0; & x < 3 \end{cases} \iff \begin{cases} x = 1; & x > 3 \\ x = 1; & x < 3 \end{cases}.$$

Note that the first alternative yielded $x = 1$, but this does not belong to the range where the equation was valid ($x > 3$) so it is not really a point where $f'(x) = 0$. However, we get $x = 1$ again in the second line (where $x < 3$) and so we do have it as a suspicious point. We need not check on signs, because we are looking for global extrema, we just check values:
— $f(1) = 4$, $f(3) = 0$.
By comparing the values of the four candidates we see that f has a global maximum $f(4) = 5$ and a global minimum $f(3) = 0$ on $[0, 4]$.

3. Because of the fraction, $D_f = (-\infty, 0) \cup (0, \infty)$. First we use the first derivative to determine the intervals of monotonicity: $f'(x) = \dfrac{e^x \cdot x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. From $f'(x) = 0$ we get a critical point $x = 1$ (using $e^x \neq 0$), we also have a critical point $x = 0$ where the derivative does not exist.

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$

We see just one local extreme, $f(1) = e$ is a local minimum.
To get the concavity stuff, we need the second derivative:

$$f''(x) = \frac{(e^x(x-1) + e^x)x^2 - e^x(x-1)2x}{x^4} = \frac{e^x(x^2 - 2x + 2)}{x^3}.$$

Since $x^2 - 2x + 2 = 0$ has no solution and $e^x \neq 0$, there is just one dividing point, $x = 0$ (from f'' not existing there). Using $e^x(x^2 - 2x + 2) > 0$ we get

	$(-\infty, 0)$	$(0, \infty)$
$f''(x)$	$-$	$+$
$f(x)$	$\frown$	$\smile$

The function changes concavity at $x = 0$, but since f is not defined at 0, $x = 0$ does not qualify as an inflection point.
Now we look at the limits at endpoints:

$$\lim_{x\to\infty}\left(\frac{e^x}{x}\right) = \left\langle\!\left\langle \tfrac{\infty}{\infty} \implies \text{LH} \right\rangle\!\right\rangle = \lim_{x\to\infty}\left(\frac{e^x}{1}\right) = \left\langle\!\left\langle e^\infty \right\rangle\!\right\rangle = \infty.$$

$$\lim_{x\to-\infty}\left(\frac{e^x}{x}\right) = \left\langle\!\left\langle \tfrac{e^{-\infty}}{\infty} = \tfrac{0}{\infty} \right\rangle\!\right\rangle = 0.$$

$$\lim_{x\to 0^+}\left(\frac{e^x}{x}\right) = \left\langle\!\left\langle \tfrac{e^0}{0^+} = \tfrac{1}{0^+} \right\rangle\!\right\rangle = \infty.$$

$$\lim_{x\to 0^-}\left(\frac{e^x}{x}\right) = \left\langle\!\left\langle \tfrac{e^0}{0^-} = \tfrac{1}{0^-} \right\rangle\!\right\rangle = -\infty.$$

To see the shape, it may be a good idea to put the above two charts together:

<table>
<tr><td>$(-\infty, 0)$</td><td>$(0, 1)$</td><td>$(1, \infty)$</td></tr>
<tr><td>$\searrow$</td><td>$\searrow$</td><td>$\nearrow$</td></tr>
<tr><td>$\frown$</td><td colspan="2">$\smile$</td></tr>
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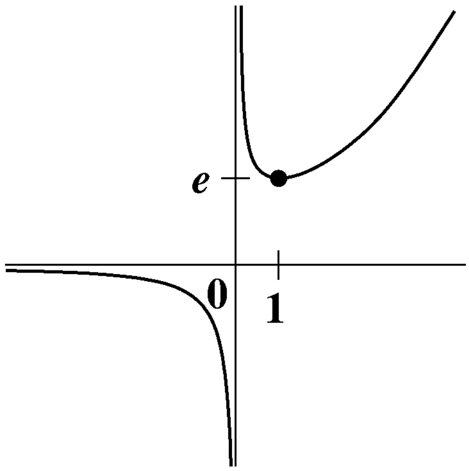


4. The domain is easy, the only problem is the fraction in the exponential. Thus $D_f = (-\infty,0)\cup(0,\infty)$. We see that there are three possibilities for asymptotes. There might be asymptotes at $\pm\infty$ and there might also be a vertical asymptote at 0. We start with this one, and to decide on it, we need to check one-sided limits at 0:

$$\lim_{x\to 0+}\bigl(x\,e^{\frac{2}{x}}\bigr) = \left\langle\!\!\left\langle 0\,e^{2/0^+} = 0\,e^{\infty} = 0\cdot\infty \implies \text{ zmna ve zlomek}\right\rangle\!\!\right\rangle = \lim_{x\to 0+}\left(\frac{e^{\frac{2}{x}}}{x^{-1}}\right)$$
$$= \left\langle\!\!\left\langle \tfrac{\infty}{\infty} \implies \text{ LH}\right\rangle\!\!\right\rangle = \lim_{x\to 0+}\left(\frac{e^{\frac{2}{x}}\frac{-2}{x^2}}{-x^{-2}}\right) = \lim_{x\to 0+}\bigl(2e^{\frac{2}{x}}\bigr) = \left\langle\!\!\left\langle 2e^{1/0^+} = 2e^{\infty} = 2\infty\right\rangle\!\!\right\rangle = \infty.$$

Since we have an infinite one-sided limit, there is a vertical asymptote at $x = 0$. It is not necessary to check the limit from the left because the vertical asymptote is already decided, but we will show it anyway as it is a nice and easy exercise:

$$\lim_{x\to 0^-}\bigl(x\,e^{\frac{2}{x}}\bigr) = 0\,e^{2/0^-} = 0\,e^{-\infty} = 0\cdot 0 = 0.$$

Note that in the first calculation (the limit from the right) there was an opportunity to simplify it; we will show it now:

$$\lim_{x\to 0+}\left(\frac{e^{\frac{2}{x}}}{x^{-1}}\right) = \left\langle\!\!\left\langle \text{substitute } y = \frac{1}{x} \implies y\to\infty\right\rangle\!\!\right\rangle = \lim_{y\to\infty}\left(\frac{e^{2y}}{y}\right)$$
$$= \left\langle\!\!\left\langle \frac{\infty}{\infty} \implies \text{ LH}\right\rangle\!\!\right\rangle = \lim_{y\to\infty}\left(\frac{2e^{2y}}{1}\right) = 2e^{\infty} = \infty.$$

Now we check whether there is an asymptote at ∞. First the limit:

$$\lim_{x\to\infty}\left(x\,e^{\frac{2}{x}}\right) = \infty\,e^{2/\infty} = \infty\,e^0 = \infty\cdot 1 = \infty.$$

This means that there is no horizontal asymptote at ∞, but the infitity leaves open the chance that there might be an oblique asymptote. To find its slope (if it exists at all) we calculate

$$k = \lim_{x\to\infty}\left(\frac{x\,e^{\frac{2}{x}}}{x}\right) = \lim_{x\to\infty}\bigl(e^{\frac{2}{x}}\bigr) = e^0 = 1.$$

Since this limit converges, there is an oblique asymptote with slope $k = 1$ at infinity. To find the shift q we use the appropriate formula:

$$q = \lim_{x\to\infty}\bigl(f(x) - k\cdot x\bigr) = \lim_{x\to\infty}\bigl(x\,e^{\frac{2}{x}} - x\bigr) = \left\langle\!\!\left\langle \infty-\infty \implies \text{ put together}\right\rangle\!\!\right\rangle$$
$$= \lim_{x\to\infty}\bigl(x(e^{\frac{2}{x}} - 1)\bigr) = \left\langle\!\!\left\langle \infty\cdot 0 \implies \text{ change into fraction}\right\rangle\!\!\right\rangle = \lim_{x\to\infty}\left(\frac{e^{\frac{2}{x}}-1}{x^{-1}}\right)$$
$$= \left\langle\!\!\left\langle \text{substitution } y = \frac{1}{x} \implies y\to 0^+\right\rangle\!\!\right\rangle = \lim_{y\to 0^+}\left(\frac{e^{2y}-1}{y}\right) = \left\langle\!\!\left\langle \frac{0}{0} \implies \text{ LH}\right\rangle\!\!\right\rangle$$
$$= \lim_{y\to 0^+}\left(\frac{2e^{2y}}{1}\right) = 2.$$

Thus there is an oblique asymptote $y = x + 2$ at infinity.
The calculations at negative infinity are similar, so we will show them briefly:

$$\lim_{x\to-\infty}\left(x\,e^{\frac{2}{x}}\right) = -\infty\,e^{-2/\infty} = -\infty\,e^0 = -\infty\cdot 1 = -\infty.$$

So no horizontal, but a chance for an oblique asymptote.

$$k = \lim_{x\to-\infty}\left(\frac{x\,e^{\frac{2}{x}}}{x}\right) = \lim_{x\to-\infty}\left(e^{\frac{2}{x}}\right) = e^0 = 1.$$

$$q = \lim_{x\to-\infty}\left(x\,e^{\frac{2}{x}} - x\right) = \lim_{x\to-\infty}\left(x(e^{\frac{2}{x}} - 1)\right) = \lim_{y\to 0^-}\left(\frac{e^{2y}-1}{y}\right) = 2.$$

The line $y = x + 2$ is also an asymptote at $-\infty$.

5a. Domain: $D_f = I\!R \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$. There is no symmetry, the function is continuous on D_f. Intercepts: $f(x) = 0 \implies x = 2$. Limits at endpoints:

$$\lim_{x\to\infty}\left(\frac{|x-2|}{x}\right) = \left\langle\!\left\langle x\to\infty \implies x > 2\right\rangle\!\right\rangle = \lim_{x\to\infty}\left(\frac{x-2}{x}\right) = 1;$$
$$\lim_{x\to-\infty}\left(\frac{|x-2|}{x}\right) = \left\langle\!\left\langle x\to-\infty \implies x < 2\right\rangle\!\right\rangle = \lim_{x\to-\infty}\left(\frac{-(x-2)}{x}\right) = -1;$$
$$\lim_{x\to 0^+}\left(\frac{|x-2|}{x}\right) = \left\langle\!\left\langle \frac{2}{0^+}\right\rangle\!\right\rangle = \infty;$$
$$\lim_{x\to 0^-}\left(\frac{|x-2|}{x}\right) = \left\langle\!\left\langle \frac{2}{0^-}\right\rangle\!\right\rangle = -\infty.$$

From the limits we see three things: There is a vertical asymptote at $x = 0$, there is a horizontal asymptote $y = 1$ at ∞ and a horizontal asymptote $y = -1$ at $-\infty$, and therefore there is no oblique asymptote at ∞ and $-\infty$.
Derivative: First we need to get rid of the absolute value:

$$f(x) = \begin{cases} \frac{x-2}{x}; & x \ge 2 \\ \frac{2-x}{x}; & x \le 2 \end{cases}.$$

Thus

$$f'(x) = \begin{cases} \frac{2}{x^2}; & x > 2 \\ -\frac{2}{x^2}; & x < 2 \end{cases}.$$

There are two critical points, $x = 0$ and $x = 2$, both coming from f' DNE. So

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$

There is a local minimum $f(2) = 0$.
Now the second derivative:

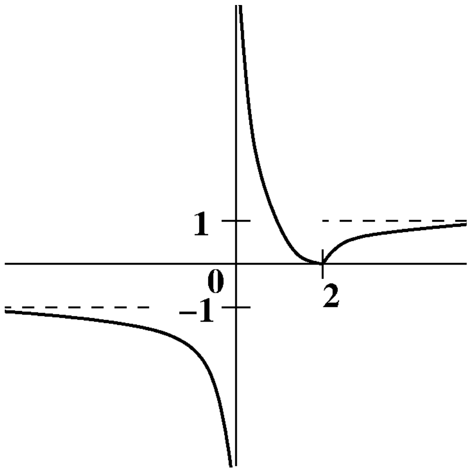
$$f''(x) = \begin{cases} -\frac{4}{x^3}; & x > 2 \\ \frac{4}{x^3}; & x < 2 \end{cases}.$$

There are two dividing points again, $x = 0$ and $x = 2$, so

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f''(x)$	$-$	$+$	$-$
$f(x)$	$\frown$	$\smile$	$\frown$

Thus $f(2) = 0$ is also an inflection point.
We put the info together and then draw the graph:

$(-\infty,0)$	$(0,2)$	$(2,\infty)$
$\searrow$	$\searrow$	$\nearrow$
$\frown$	$\smile$	$\frown$



5b. Domain: $D_f = \mathbb{R}$, since $e^{3x}+1 \geq 1 > 0$ always; this means that there are no vertical asymptotes. There is no symmetry, the function is continuous on D_f. Intercepts: $f(0)=\ln(2)$. Limits at endpoints:

$$\lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big)\Big) = \Big\langle\!\Big\langle \ln\big(e^{\infty}+1\big) = \ln(\infty)\Big\rangle\!\Big\rangle = \infty;$$

$$\lim_{x\to-\infty}\Big(\ln\big(e^{3x}+1\big)\Big) = \Big\langle\!\Big\langle \ln\big(e^{-\infty}+1\big) = \ln(0+1)\Big\rangle\!\Big\rangle = 0.$$

Thus there is a horizontal asymptote $y=0$ at $-\infty$. There is no horizontal symptote at ∞ (there might be an oblique asymptote there).
Derivative:

$$f'(x) = \frac{3e^{3x}}{e^{3x}+1}.$$

This derivative exists everywhere and is always positive (as $e^{3x} > 0$), so there are no critical points (hence no local extrema) and the function f is increasing on $\mathbb{R}$.
Now the second derivative:

$$f''(x) = \frac{9e^{3x}}{(e^{3x}+1)^2}.$$

Again, the second derivative exists everywhere and is always positive, so there are no inflection points and the function f is concave up on $\mathbb{R}$.
So the function is always increasing and concave up. It remains to check on an oblique asymptote at ∞. First:

$$k = \lim_{x\to\infty}\big(f'(x)\big) = \lim_{x\to\infty}\Big(\frac{3e^{3x}}{e^{3x}+1}\Big) = 3.$$

Since this limit converges, there is an oblique asymptote at ∞. We find q:

$$q = \lim_{x\to\infty}\big(f(x)-kx\big) = \lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big)-3x\Big) = \Big\langle\!\Big\langle \infty-\infty \implies \text{ put together}\Big\rangle\!\Big\rangle$$

$$= \lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big)-\ln\big(e^{3x}\big)\Big) = \lim_{x\to\infty}\Big(\ln\Big(\frac{e^{3x}+1}{e^{3x}}\Big)\Big) = \ln(1) = 0.$$

Thus $y=3x$ is an oblique asymptote at ∞. Graph: