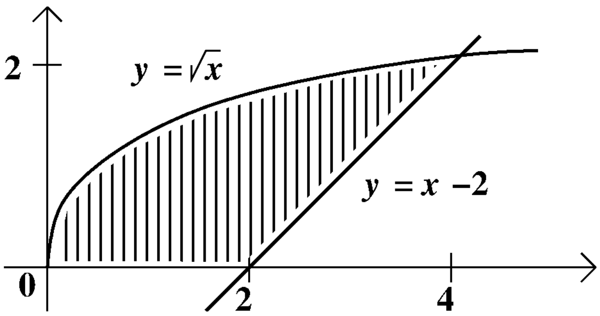
Calculus 1 Integral–applications: Solution

Note: The parts between pairs of $\langle\!\langle\ \rangle\!\rangle$ are explanatory notes. These are not "official" parts of the solution, they merely illustrate how we think about the problem. When writing an "official" solution of a problem, such parts should be left out.

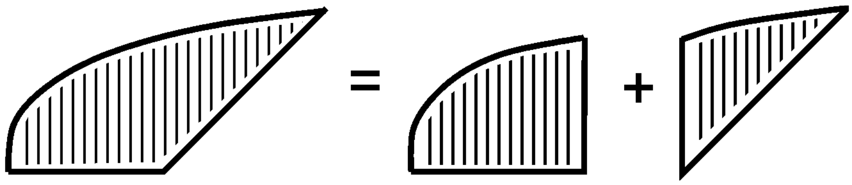
1. Since $f(x) = \dfrac{1}{\sqrt{1-x^2}} > 0$ on the interval $\left[0, \frac{1}{2}\right]$, the area is simply the appropriate definite integral. To integrate the given function we recall that it is in fact an elementary integral:

$$A = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \Big[\arcsin(x)\Big]_0^{1/2} = \arcsin\left(\tfrac{1}{2}\right) - \arcsin(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

2. First we have to draw a picture:



We see that our region is not of the basic type, therefore we have to split it into two pieces:



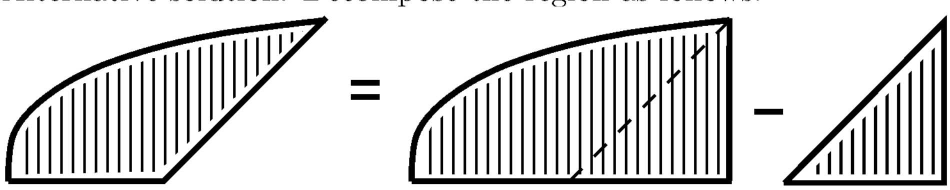
The areas are:

$$A_1 = \int_0^2 \sqrt{x} - 0\,dx = \left[\frac{2}{3}\sqrt{x^3}\right]_0^2 = \frac{2}{3}\sqrt{2^3} - 0 = \frac{4}{3}\sqrt{2}.$$
$$A_2 = \int_2^4 \sqrt{x} - (x-2)\,dx = \left[\frac{2}{3}\sqrt{x^3} - \frac{1}{2}x^2 + 2x\right]_2^4 = \frac{10}{3} - \frac{4}{3}\sqrt{2}.$$

Therefore the area is

$$A = A_1 + A_2 = \frac{10}{3}.$$

Alternative solution: Decompose the region as follows:



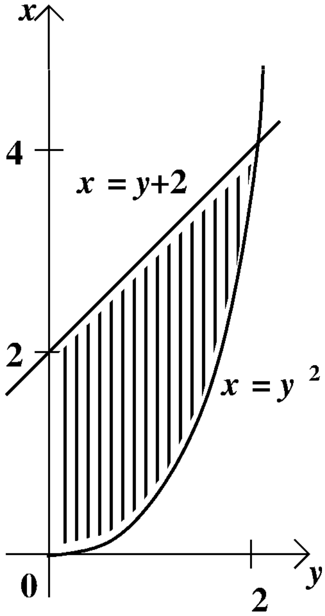
Now

$$B_1 = \int_0^4 \sqrt{x}\,dx = \left[\frac{2}{3}\sqrt{x^3}\right]_0^4 = \frac{16}{3}.$$
$$B_2 = \int_2^4 x - 2\,dx = \left[\frac{1}{2}x^2 - 2x\right]_2^4 = 2.$$

Thus

$$A = B_1 - B_2 = \frac{10}{3}.$$

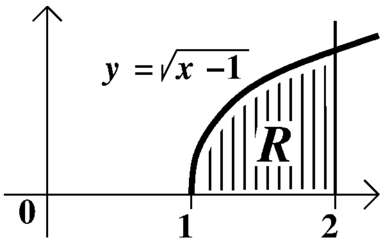
Alternative solution: It is also possible to switch axes and pass to inverse functions:



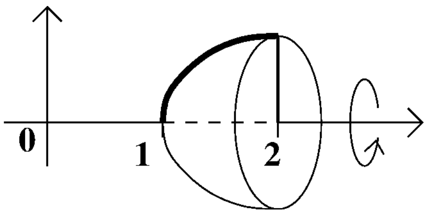
Now we can find the area with just one integral:

$$A=\int_0^2 (y+2)-y^2\,dy=\Big[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big]_0^2=\frac{10}{3}.$$

3. First draw a picture and identify the region R:

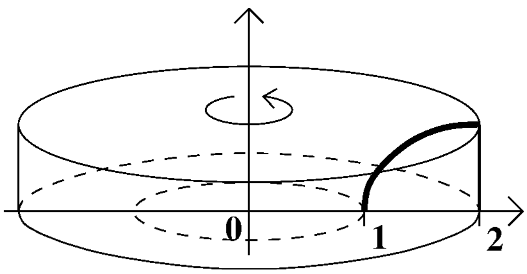


a) When this region is revolved about the x-axis, the vertical slices form discs, therefore we use the disc method (or we realize that we have a horizontal axis, which means the disc method):



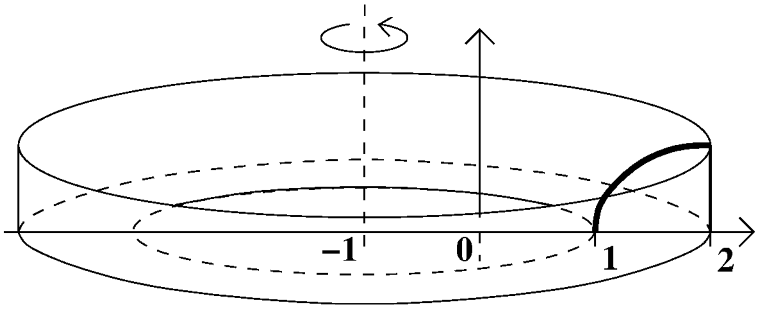
$$V_a=\pi\int_1^2 \sqrt{x^2-1}^{\,2}\,dx=\pi\int_1^2 x^2-1\,dx=\pi\Big[\frac{1}{3}x^3-x\Big]_1^2=\frac{4}{3}\pi.$$

b) Now vertical slices form shells, so we have to use the shell method (or we see it from the fact that the asix of rotation is vertical):



$$V_b=2\pi\int_1^2 x\sqrt{x^2-1}\,dx=\left|\begin{matrix} y=x^2-1\\ dy=2x\,dx\end{matrix}\right|=\pi\int_0^3\sqrt{y}\,dy=\pi\Big[\frac{2}{3}\sqrt{y^3}\Big]_0^3=2\sqrt{3}\pi.$$

c) We have a vertical axis of rotation again, so we use the shell method, but now with a shifted axis of rotation:



$$V_c = 2\pi\int_1^2 (x+1)\sqrt{x^2-1}\,dx = 2\pi\int_1^2 x\sqrt{x^2-1}\,dx + 2\pi\int_1^2 \sqrt{x^2-1}\,dx = I_1 + I_2.$$

The first integral is just like in part b) and we can copy the answer. The second integral is of the type "root of quadratics" and we use the appropriate indirect substitution.

$$I_2 = \left|\begin{array}{c} x = \frac{1}{\cos(t)} \\ dx = \frac{\sin(t)\,dt}{\cos^2(t)} \\ x=1 \mapsto \cos(t)=1 \mapsto t=0 \\ x=2 \mapsto \cos(t)=\frac{1}{2} \mapsto t=\pi/3 \end{array}\right| = 2\pi\int_0^{\pi/3} \sqrt{\frac{1}{\cos^2(t)} - 1}\frac{\sin(t)\,dt}{\cos^2(t)}$$

$$= 2\pi\int_0^{\pi/3} \sqrt{\frac{1-\cos^2(t)}{\cos^2(t)}}\frac{\sin(t)\,dt}{\cos^2(t)} = 2\pi\int_0^{\pi/3} \sqrt{\frac{\sin^2(t)}{\cos^2(t)}}\frac{\sin(t)\,dt}{\cos^2(t)}$$

$$= 2\pi\int_0^{\pi/3} \frac{\sin(t)}{\cos(t)}\frac{\sin(t)\,dt}{\cos^2(t)} = 2\pi\int_0^{\pi/3} \frac{\sin^2(t)\,dt}{\cos^3(t)}.$$

This is a trigonometric integral. There is an "extra cosine", so we move it to the top and prepare the integral for the $y = \sin(t)$ substitution.

$$I_2 = 2\pi\int_0^{\pi/3} \frac{\sin^2(t)}{\cos^4(t)}\cos(t)\,dt = 2\pi\int_0^{\pi/3} \frac{\sin^2(t)}{[1-\sin^2(t)]^2}\cos(t)\,dt$$

$$= \left|\begin{array}{c} y = \sin(t) \\ dy = \cos(t)\,dt \\ t=0 \mapsto y=0 \\ t=\frac{\pi}{3} \mapsto y=\frac{\sqrt{3}}{2} \end{array}\right| = 2\pi\int_0^{\sqrt{3}/2} \frac{y^2}{[1-y^2]^2}\,dy = \pi\int_0^{\sqrt{3}/2} \frac{2y^2}{(1-y)^2(1+y)^2}\,dy.$$

We obtained an integral from a rational function, which calls for partial fractions:

$$I_2 = \pi\int_0^{\sqrt{3}/2} \frac{2y^2}{(1-y)^2(1+y)^2}\,dy$$

$$= \pi\int_0^{\sqrt{3}/2} \frac{\frac{1}{2}}{(1-y)^2} - \frac{\frac{1}{2}}{1-y} + \frac{\frac{1}{2}}{(1+y)^2} - \frac{\frac{1}{2}}{1+y}\,dy$$

$$= \pi\Big[\frac{1}{2}\frac{1}{1-y} - \frac{1}{2}\frac{1}{1+y} + \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y|\Big]_0^{\sqrt{3}/2}$$

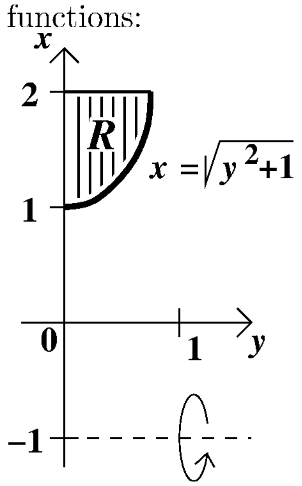
$$= \pi\Big[\frac{y}{1-y^2} + \frac{1}{2}\ln\Big|\frac{1-y}{1+y}\Big|\Big]_0^{\sqrt{3}/2} = 2\sqrt{3}\pi + \frac{\pi}{2}\ln\Big|\frac{2-\sqrt{3}}{2+\sqrt{3}}\Big| = 2\sqrt{3}\pi + \frac{\pi}{2}\ln\Big|\frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})}\Big|$$

$$= 2\sqrt{3}\pi + \frac{\pi}{2}\ln|7-4\sqrt{3}|.$$

Thus

$$V_c = I_1 + I_2 = 2\sqrt{3}\pi + 2\sqrt{3}\pi + \frac{\pi}{2}\ln|7-4\sqrt{3}| = 4\sqrt{3}\pi + \frac{\pi}{2}\ln|7-4\sqrt{3}|.$$

Alternative solution for the difficult part c): We try to switch the axes and pass to inverse functions:



We now have to use the disc method:

$$V_c = \pi\int_0^{\sqrt{3}} 3^2 - \left(1+\sqrt{y^2+1}\right)^2 dy = \pi\int_0^{\sqrt{3}} 7 - y^2 - 2\sqrt{y^2+1}\,dy.$$

Again we have a root of a quadratic expression there and so the calculations are most likely equally nasty.

4. We need to find the length of a piece of a parabola. We have a formula for that:

$$L = \int_{-1}^{1}\sqrt{1+([x^2]')^2}\,dx = \int_{-1}^{1}\sqrt{1+(2x)^2}\,dx = \int_{-1}^{1}\sqrt{1+4x^2}\,dx.$$

This integral is of the type "root of qudratics" and we apply the appropriate substitution:

$$L = \left|\begin{matrix} 2x = \tan(t) \\ 2\,dx = \frac{1}{\cos^2(t)}\,dt \end{matrix}\right| = \int_{x=-1}^{x=1}\sqrt{1+\frac{\sin^2(t)}{\cos^2(t)}}\frac{dt}{2\cos^2(t)}$$

$$= \int_{x=-1}^{x=1}\sqrt{\frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)}}\frac{dt}{2\cos^2(t)} = \frac{1}{2}\int_{x=-1}^{x=1}\frac{dt}{\cos^3(t)} = \frac{1}{2}\int_{x=-1}^{x=1}\frac{\cos(t)\,dt}{\cos^4(t)}.$$

We obtained a trigonometric integral with an "extra cosine", so in the last step we moved it into the numerator to prepare the integral for a sine substitution. Since the procedure that follows is similar to the previous solution, we prefer to try a different way, especially since the previous solution was rather long. We start with the second-last expression and write it using secant, then we use reduction formulas.
Note that we did not change the limits to the language of t's. The reason is that we would have to find some angles from $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangents are $\pm\frac{1}{2}$ and that is rather ugly. We prefer to evaluate the indefinite integral first, then do a back substitution to get from t's to x's and finally substitute the proper limits. But now we look at that secant problem:

$$\int\frac{dt}{\cos^3(t)} = \int\sec^3(t)\,dt = \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\int\sec(t)\,dt$$

$$= \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln|\tan(t)+\sec(t)| + C,\ x \neq \frac{\pi}{2} + k\pi.$$

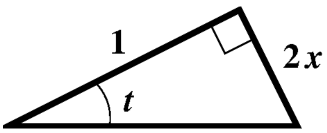
To do the back substitution we need to know what to substitute for $\sec(t)$ (we already know that for $\tan(t)$ we substitute $2x$). We try to find some way to express secant using tangent:

$$\sec(t) = \sqrt{\frac{1}{\cos^2(t)}} = \sqrt{\frac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}} = \sqrt{\tan^2(t)+1} = \sqrt{(2x)^2+1}.$$

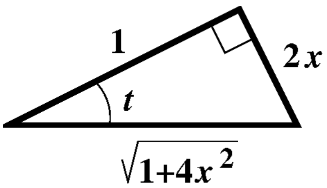
Thus we have

$$\begin{aligned} L &= \frac{1}{2}\Big[\frac{1}{2}\sqrt{1+4x^2}\cdot 2x + \frac{1}{2}\ln\big|2x+\sqrt{1+4x^2}\big|\Big]_{x=-1}^{x=1} \\ &= \frac{1}{2}\Big(\sqrt{5}+\frac{1}{2}\ln|2+\sqrt{5}| + \sqrt{5} - \frac{1}{2}\ln|-2+\sqrt{5}|\Big) = \sqrt{5}+\frac{1}{4}\ln\Big|\frac{2+\sqrt{5}}{-2+\sqrt{5}}\Big| \\ &= \sqrt{5}+\frac{1}{4}\ln\Big|\frac{(2+\sqrt{5})^2}{(\sqrt{5}-2)(\sqrt{5}+2)}\Big| = \sqrt{5}+\frac{1}{2}\ln|2+\sqrt{5}|. \end{aligned}$$

Note on the back substitution: Instead of the algebra, one can try to use an appropriate triangle. First of all, the substitution $\tan(t) = 2x = \frac{2x}{1}$ corresponds to this triangle:



We fill in the third side using the Pythagoras rule:



Now we immediately see that $\cos(t) = \dfrac{1}{\sqrt{1+4x^2}}$ and hence $\sec(t) = \frac{1}{\cos(t)} = \sqrt{1+4x^2}$.

5. Since each limit of the integral leads to trouble (infinity is always a problem, and 1 gives zero in the denominator), we have to split the integral into two, we pick some nice number as a dividing point. Then we use limits to handle the troublesome points:

$$\begin{aligned} I &= \int_1^\infty \frac{dx}{x\sqrt{x-1}} = \int_1^2 \frac{dx}{x\sqrt{x-1}} + \int_2^\infty \frac{dx}{x\sqrt{x-1}} \\ &= \lim_{a\to 1^+}\Big(\int_a^2 \frac{dx}{x\sqrt{x-1}}\Big) + \lim_{b\to\infty}\Big(\int_2^b \frac{dx}{x\sqrt{x-1}}\Big). \end{aligned}$$

To save time, we first find the indefinite integral using the standard substitution for getting rid of a square root.

$$\begin{aligned} \int \frac{dx}{x\sqrt{x-1}} &= \left|\begin{aligned} x-1 &= y^2 \\ dx &= 2y\,dy \\ x &= y^2+1 \end{aligned}\right| = \int \frac{2y\,dy}{(y^2+1)y} = 2\int \frac{dy}{y^2+1} = 2\arctan(y)+C \\ &= 2\arctan(\sqrt{x-1})+C,\ x>1. \end{aligned}$$

Now we can try to evaluate the integral:

$$\begin{aligned} I &= \lim_{a\to 1^+}\Big[2\arctan(\sqrt{x-1})\Big]_a^2 + \lim_{b\to\infty}\Big[2\arctan(\sqrt{x-1})\Big]_2^b \\ &= 2\arctan(1) - 2\lim_{a\to 1^+}\big(\arctan(\sqrt{a-1})\big) + 2\lim_{b\to\infty}\big(\arctan(\sqrt{b-1})\big) - 2\arctan(1) \\ &= 0 + 2\cdot\frac{\pi}{2} = \pi. \end{aligned}$$

We see that the integral is convergent (both parts converged) and the answer is π.

6. We could actually evaluate this integral using a partial fractions decomposition, but that would be too much work. It is easier to use some test. We notice that there seem to be no easy way to use the Comparison Test, so we try the Limit Comparison Test (LCT). First we guess a test function. When x is really big, we can ignore lower powers and get comparison

$$f(x) = \frac{x-1}{x^2-x+1} \sim \frac{x}{x^2} = \frac{1}{x} = g(x).$$

We have to check that our guess was correct by comparing these two functions near infinity:

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\infty}\left(\frac{\frac{x-1}{x^2-x+1}}{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{x^2-x}{x^2-x+1}\right) = 1.$$

Since this limit converged, and to a number not equal to zero, the comparison is valid and we can now compare the corresponding integrals. We know that $\int_2^\infty \frac{dx}{x}$ diverges, therefore by LCT, also the given integral $\int_2^\infty \frac{x-1}{x^2-x+1}\,dx$ diverges.